PARASOLUBILITY

IN

LIE RINGS AND LIE ALGEBRAS

by

S.G.BRAZIER

A thesis submitted for Ph.D. at the University of Warwick

1974

ABSTRACT

Parasoluble groups have been defined by Wehrfritz as a generalization of nilpotent and supersoluble groups, and this thesis is concerned with the analogous Lie rings and Lie algebras. Most results are proved for Lie rings only, but in several cases the proofs for general Lie algebras are identical.

Chapter 1 sets up notation and terminology and introduces the concepts of power derivations and quasi-centralizers.

Chapter 2 defines the paralyzer and stabilizer of a series as terms similar to those used in group theory. Also defined are quasicentral series, and parasoluble Lie rings as those with a finite quasicentral series.

In chapter 3 it is shewn that under certain conditions the paralyzer of a finite series is parasoluble. This is always true for torsion-free Lie rings, but an example is given to shew that it is not true in general.

Chapter 4 is concerned with paralyzers of ascending series and results are obtained which are generalizations of Lie ring analogues of some results of Hall and Hartley.

Chapter 5 looks at the join problem for hypercyclic, parasoluble and supersoluble Lie rings.

Chapter 6 is concerned with the class of soluble Lie rings in which all subideals are ideals. These are the Lie ring analogue of simlarly defined groups of Robinson.

Chapter 7 deals with local parasolubility and we shew that Lie rings which locally have a quasicentral series of bounded length are parasoluble.

Chapter 8 employs some of the methods of the preceeding chapters to obtain group-teoretic results, the main one being an improvement of a theorem of Hill.

I would like to thank my supervisor, Dr I.N.Stewart for his suggestions and help in this work, particularly in chapters 7 and 8. I am also grateful to the Science Research Council for their financial support.

CONTENTS

STANDARD NOTATION

Z	denotes the integers
Z^*	the non-zero integers
Z^+	the positive integers
Z_n	the integers modulo n
Z_{p^∞}	the quasicyclic p-group
$Z_{\bar{p}}$	the p-adic integers
Q	the rational numbers
S_n	the symmetric group on n letters
Q_8	the quaternion group
$M \otimes_R N$	the tensor product over R of M and N
$n \mid m$	n divides m
$n \nmid m$	n does not divide m
h.c.f.(m,n)	the highest common factor of m and n
$\lvert S\rvert$	the cardinality of the set S
ω	the first infinite ordinal

CHAPTER 1

Basic Definitions

Let L be an abelian group (written additively) then L is said to be a Lie ring if there is a bilinear multiplication $(x,y) \to [x,y]$ defined on L such that:

(i) $[x,x] = 0$ for all $x \in L$

(ii) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ for all $x,y,z \in L$

(This is known as the Jacobi identity)

L is an abelian Lie ring if $[x,y] = 0$ for all $x,y \in L$.
More generally we shall denote by L^* the abelian Lie ring whose underlying additive group is L.

Let X be a subset of L then:

(1) X is a subring of L, written $X \leq L$ if $x \pm y$, $[x,y] \in X$ for all $x,y \in X$.

(2) X is a subgroup of L if $X^* \leq L^*$, i.e. $x \pm y \in X$ if $x,y \in X$.

(3) X is an ideal of L, written $X \lhd L$, if X is a subgroup of L and $[x,y] \in X$ for all $x \in X$ and $y \in L$.

(4) If $X \lhd L$ the quotient ring L/X is the set of cosets $\{X+y : y \in L\}$ which is a Lie ring under the operations:

(a) $(X+y) + (X+z) = X+(y+z)$

(b) $[X+y, X+z] = X+[y,z]$

(5) X is a subideal of L, written X si L if there are subrings $X_0, X_1, \dots, X_n$ of L such that:

$$X = X_0 \lhd X_1 \lhd \dots \lhd X_n = L.$$

Notation

Let $X,Y,\ldots,W$ be subsets of L and $x,y,\ldots,w \in L$

(1) We write $[x,y,\ldots,w]$ for $[[\ldots[x,y],\ldots],w]$

(2) $[X,Y,\ldots,W]$ denotes the set $\{[x,y,\ldots,w] : x \in X,\ y \in Y,\ w \in W\}$

(3) $[X,_nY]$ denotes $[X,\underbrace{Y,\ldots,Y}_{n}]$ and $[x,_ny] = [x,\underbrace{y,\ldots,y}_{n}]$

(4) $X + Y = \{x + y : x \in X,\ y \in Y\}$

(5) X^Y denotes the smallest subgroup of L containing X and which is invariant under multiplication by elements of Y. Thus X^Y is the subgroup generated by $X \cup \sum_n [X,_nY]$.

(6) $< X >$ is the subring generated by the elements of X. In particular $< x > = \{ nx : n \in Z \}$.

Isomorphisms

If L and M are Lie rings and $\theta : L \to M$ such that for all $x,y \in L$:

$$(1)\ \theta(x+y) = \theta(x) + \theta(y)$$

$$(2)\ \theta[x,y] = [\theta(x),\theta(y)]$$

then θ is a Lie homomorphism.

If also $\ker\theta = 0$ and $\operatorname{im}\theta = M$ then θ is an isomorphism between L and M, and we write $L \cong M$.

If $H \lhd L$ and $K \leqslant L$ it is easy to show that:

$$(H + K)/H \cong K/(K \cap H).$$

Direct Sums

Suppose L is a Lie ring and $L_\sigma \leqslant L$ for all $\sigma \in \Lambda$ and

(1) $L_\tau \cap \bigcap_{\substack{\sigma\in\Lambda \\ \sigma\neq\tau}} L_\sigma = 0$

(2) $[L_\sigma, L_\tau] = 0$ if $\sigma \neq \tau$

(3) $\sum_{\sigma\in\Lambda} L_\sigma = L$

then L is the <u>direct sum</u> of the L_σ and we write $L = \coprod_{\sigma\in\Lambda} L_\sigma$

Then if $x \in L$ there exist $\sigma_1, \ldots, \sigma_n \in \Lambda$ such that x may be written uniquely in the form $x = x_1 + \ldots + x_n$ where $x_i \in L_{\sigma_i}$ for $1 \leqslant i \leqslant n$.

If $\Lambda = \{1,2\}$, we shall write $L_1 \oplus L_2$ for the direct sum of L_1 and L_2.

Torsion Properties

We use the standard terminology from abelian group theory for the terms: order, periodic, torsion, torsion-free, p-element, exponent, and endomorphism, as applied to the underlying abelian group of a Lie ring L. We denote the set of endomorphisms of L by End(L), the exponent of L by exp(L) and the order of an element $x \in L$ by $|x|$.

The <u>adjoint</u> map $x^* : L \to L$ given by $yx^* = [y,x]$ where $x,y \in L$ is clearly an endomorphism of L, so $|[y,x]|$ divides $|y|$ and since $[x,y] = -[y,x]$ it also divides $|x|$ ...(i).

Now the set of torsion elements T of L clearly forms a subgroup of L and (i) shows it is an ideal, i.e. $T \lhd L$.

If T_p denotes the set of p-elements of T and $x \in T_p$ and $y \in T_q$ where $q \neq p$ then (i) shows that $[y,x] = 0$, i.e. $[T_p,T_q] = 0$. Hence T is the direct sum $\coprod_p T_p$.

Also if $X \leq T$ and $X = \langle x_1,x_2,\ldots,x_n \rangle$ then the exponent of X, exp(X) is finite.

Classes of Lie rings

In a similar way to that of Hall [5], we define classes of Lie ring. A class $\underline{\underline{X}}$ of Lie rings will denote a class in the usual sense, whose members are Lie rings, and with the additional properties:

(1) $0 \in \underline{\underline{X}}$ where 0 is a trivial Lie ring

(2) $K \cong L \in \underline{\underline{X}}$ implies that $K \in \underline{\underline{X}}$.

We define the product of two classes $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ by saying that $L \in \underline{\underline{XY}}$ if and only if there exists $I \lhd L$ such that $I \in \underline{\underline{X}}$ and $L/I \in \underline{\underline{Y}}$. We write $\underline{\underline{X}}^2$ for $\underline{\underline{XX}}$. We reserve the following notation for the more familiar classes:

(i) $L \in \underline{\underline{C}}$ if and only if there exists $x \in L$ with $\langle x \rangle = L$.
$\underline{\underline{C}}$ is the class of cyclic Lie rings.

(ii) $L \in \underline{\underline{A}}$ if and only if $[x,y] = 0$ for all $x,y \in L$.
$\underline{\underline{A}}$ is the class of abelian Lie rings.

(iii) $L \in \underline{\underline{G}}_n$ if and only if there exist $x_1,x_2,\ldots,x_n \in L$ with $L = \langle x_1,x_2,\ldots,x_n \rangle$.

$\underline{\underline{G}} = \bigcup_{n=1}^{\infty} \underline{\underline{G}}_n$ is the class of finitely-generated Lie rings.

(iv) $L \in \underline{\underline{\Gamma}}'_n$ if and only if $L^x \in \underline{\underline{G}}_n$. $\underline{\underline{\Gamma}} = \bigcup_{n=1}^{\infty} \underline{\underline{\Gamma}}_n$

(v) $L \in \underline{\underline{F}}$ if and only if $|L| < \infty$.

$\underline{\underline{F}}$ is the class of finite Lie rings.

<u>Closure Operations</u>

A closure operation A asssigns to each class $\underline{\underline{X}}$ another class $A\underline{\underline{X}}$ such that for all classes $\underline{\underline{X}}$ and $\underline{\underline{Y}}$ the following axioms are satisfied:

(1) $A(0) = (0)$ where (0) is the class of trivial Lie rings.

(2) $\underline{\underline{X}} \leqslant A\underline{\underline{X}}$

(3) $A(A\underline{\underline{X}}) = A\underline{\underline{X}}$

(4) $\underline{\underline{X}} \leqslant \underline{\underline{Y}}$ implies that $A\underline{\underline{X}} \leqslant A\underline{\underline{Y}}$

$\underline{\underline{X}}$ is said to be A-closed if $\underline{\underline{X}} = A\underline{\underline{X}}$

We define as follows the operations S,L,R,Q,P:

(i) $L \in S\underline{\underline{X}}$ if and only if $L \cong K \leqslant M \in \underline{\underline{X}}$

(ii) $L \in L\underline{\underline{X}}$ if and only if $K \leqslant L$, $K \in \underline{\underline{G}}$ implies that $K \in S\underline{\underline{X}}$

(iii) $L \in R\underline{\underline{X}}$ if and only if for each $x \in L$ there exists $I_x \lhd L$ such that $x \notin I_x$ and $L/I_x \in \underline{\underline{X}}$

(iv) $L \in Q\underline{\underline{X}}$ if and only if there exists $K \lhd M \in \underline{\underline{X}}$ with $L \cong M/K$

(v) $L \in P\underline{\underline{X}}$ if and only if there exist $0 = L_0 \lhd L_1 \lhd \ldots \lhd L_n = L$ such that $L_i/L_{i-1} \in \underline{\underline{X}}$ for all $1 \leqslant i \leqslant n$.

$P\underline{\underline{A}}$ is known as the class of <u>soluble</u> Lie rings.

Modules

If L is a Lie ring, V is an <u>L-module</u> if V is an abelian group (under addition) admitting a (right) multiplication by elements of L such that for all $x,y \in L$ and $u,v \in V$ the following axioms are satisfied:

$$(1)\quad (u+v)x = ux + vx$$

$$(2)\quad u(x+y) = ux + uy$$

$$(3)\quad u[x,y] = (ux)y - (uy)x$$

If V is also a Lie ring we may form the <u>semi-direct sum</u> $V + L$ as a Lie ring by defining $[v,x] = vx$. Thus if $I \triangleleft L$ then I is an L-module, since (3) is implied by the Jacobi identity. Note also that I^* is an L-module.

For convenience we shall assume that every module has an abelian Lie structure, unless specified otherwise.

We shall say that W is an <u>L-submodule</u> of V, if W is an L-invariant subring of V.

Derivations

Let $d : L \to L$ such that for all $x,y \in L$:

$$(1)\quad (x+y)d = xd + yd$$

$$(2)\quad [x,y]d = [xd,y] + [x,yd]$$

then d is called a <u>derivation</u> of L, and the set of all derivations of L is denoted by $\mathrm{Der}(L)$. If $d,e \in \mathrm{Der}(L)$ and we define $[d,e] = de - ed$, $\mathrm{Der}(L)$ becomes a Lie ring, and L is then a $\mathrm{Der}(L)$-module. We may therefore form $L + \mathrm{Der}(L)$

as a Lie ring by defining $[x,d] = xd$ if $x \in L$.

From (1) d is an endomorphism of L so by (2), if L is abelian, every endomorphism of L is then a derivation. Hence $Der(L^*) = End(L)$, the set of endomorphisms of L, in general.

The map $x^* : L \to L$ given by $x^* ; y \to [y,x]$ where $x,y \in L$ is a derivation of L, called an <u>inner derivation</u>.

We define $I \leqslant L$ to be a <u>characteristic</u> ideal of L, and write I ch L, if $Id \leqslant I$ for all the derivations d of L. Then if I ch $L \lhd M$ then $I \lhd M$ since the inner derivations of M restricted to L are derivations of L.

<u>Power derivations</u>

We define $d \in Der(L)$ to be a power derivation of L if for every $X \leqslant L$, $Xd \leqslant X$. Thus if $x \in L$, $xd \in \langle x \rangle$ so $xd = nx$ for some $n \in Z$. If e is also a power derivation of L and $xe = mx$ for some $m \in Z$ then $x[e,d] = xed - xde = nmx - mnx = 0$. Hence the set of power derivations forms an abelian Lie ring. In general n defined above depends on x and d. If n can be chosen independently of x then d willbe said to be <u>universal</u> on L.

<u>Lemma 1.1</u>

If d is a power derivation of L and L has a torsion-free element, then d is universal on L.

<u>Proof</u>

Suppose $x,y \in L$ where $|x| = \infty$. There exist $n,m,k \in Z$ such that $xd = nx$, $yd = my$, and $(x+y)d = k(x+y)$. So $(n-k)x = (k-m)y$ and $(n-k)xd = (k-m)yd$ i.e. $(n-k)nx = (k-m)my$. Therefore $(n-k)(m-n)x = 0$. Thus either $n = m$ or $n = k$, and $my = ny$.

We therefore turn our attention now to periodic Lie rings and assume that L is such a ring. First we look at L_p, the set of p-elements of L. Let $x,y \in L_p$ where $|y| = p^r$ and $|x| = p^t \leqslant p^r$. Now there exists $z \in L_p$ such that $x \in \langle y\rangle + \langle z\rangle$ and $|z| = p^s \leqslant p^t$. If d is a power derivation of L then there exist $n,m,k \in Z$ such that $zd = nz$, $yd = my$, and $(y+z)d = k(y+z)$. Therefore $my = ky$ i.e. $k \equiv m \bmod p^r$ and $nz = kz$ i.e. $k \equiv n \bmod p^s$. Thus $m \equiv n \bmod p^s$ so $zd = mz$ and $xd = mx$. Hence if $X \leqslant L_p$ and X has finite exponent p^r then d is universal on X and $xd = mx$ for all $x \in X$ and $0 \leqslant m < p^r$. In general there corresponds to d a p-adic integer $\bar{r} = r_0 + r_1p + r_2p^2 + \ldots + r_jp^j + \ldots$ such that if $x \in L_p$ has order p^j then $xd = (r_0 + r_1p + \ldots + r_{j-1}p^{j-1})x$. $\bar{r}$ is uniquely defined if $\exp(L_p) = \infty$, otherwise defined uniquely modulo $\exp(L_p)$. (The above argument is taken from [4] P. 210.)

Now we return to L and suppose that $Y \leqslant L$ and $\exp(Y)$ is finite, say $p_1^{j_1}p_2^{j_2}\ldots p_s^{j_s}$. Then $\exp(Y_{p_i}) = p_i^{j_i}$ if Y_{p_i} is the set of p_i-elements of Y, and if $y \in Y$ then $y = y_1 + y_2 + \ldots + y_s$ where $y_i \in Y_{p_i}$ for $1 \leqslant i \leqslant s$. Suppose $y_id = m_iy_i$ where $0 \leqslant m_i \leqslant p_i^{j_i}$ and is independent of y_i. Then $yd = \sum_{i=1}^{s} y_id = \sum_{i=1}^{s} m_iy_i$.

By the Chinese remainder theorem (e.g. [11] p.63) there exists $n \in Z$ such that $n = m_i \bmod p_i^{j_i}$ for all $1 \leqslant i \leqslant s$. Therefore $yd = \sum_{i=1}^{s} ny_i = ny$. This proves:

Lemma 1.2

Power derivations are universal on Lie rings of finite exponent.

Corollary 1.2.1

Power derivations are universal on $\underline{\underline{G}}$-Lie rings.

Proof

Suppose $L \in \underline{\underline{G}}$ and d is a power derivation of L. If L is not periodic, d is universal by lemma 1.1. If L is periodic then since $L \in \underline{\underline{G}}$, L has finite exponent so d is universal by lemma 1.2.

Lemma 1.3

If L is a Lie ring and d is a non-trivial power derivation of L then $[L,L]d = 0$. If also $[L,L]$ is torsion-free then $L \in \underline{\underline{A}}$.

Proof

We can assume $L \notin \underline{\underline{A}}$ and choose $x,y \in L$ such that $[x,y] \neq 0$. Since $< x,y > \in \underline{\underline{G}}$, by corollary 1.2.1 there exists $n \in Z^*$ such that $xd = nx$, $yd = ny$, and $[x,y]d = n[x,y]$. But d is a derivation so $[x,y]d = [xd,y] + [x,yd] = 2n[x,y]$. So $n[x,y] = 0$ and $[x,y]d = 0$ i.e. $[L,L]d = 0$. If also $[L,L]$ is torsion-free then $[x,y] = 0$ since $n \neq 0$, so $[L,L] = 0$ and $L \in \underline{\underline{A}}$.

Example

Let $L = \langle x,y : [x,y] = x,\ p^2y = px = 0 \rangle$ and d the map $d : L \to L$ such that $zd = pz$ for all $z \in L$. Then since $[x,y]d$ is zero, d is clearly a power derivation of L. Therefore L is a non-abelian Lie ring with a non-trivial power derivation.

Lemma 1.4

If L is a Lie ring of which every inner derivation is a power derivation (equivalently every subring is an ideal) then $L \in \underline{\underline{A}}$.

Proof

Let $x,y \in L$; we shall show that $[x,y] = 0$. Since $\langle x,y \rangle \in \underline{\underline{G}}$ by corollary 1.2.1 there exists $n \in Z$ such that $[y,x] = ny$ and $[x,x] = nx = 0$. Therefore if x is torsion-free, $[y,x] = 0$. Now suppose x is periodic; w.l.o.g. we may assume x has order p^s for some prime p. Since x annihilates all q-elements of L if q is a prime different from p, and $[y,x] = 0$ if y is torsion-free, we may also assume y has order p^t. Hence n is a multiple of p^s, say mp^s. Then $[y,x] = mp^sy$ so if $t \leqslant s$ then $[y,x] = 0$. If $s \leqslant t$ then similarly there exists $r \in Z$ such that $[x,y] = rp^tx = 0$. Hence $L \in \underline{\underline{A}}$.

This result is proved in [1] by a different method.

The Quasicentralizer

Let $X^* \leqslant Y^* \leqslant M^*$ where M is an L-module; we define:

(1) the centralizer of Y mod X in L

$C = C_L(Y/X) = \{ x \in L : [y,x] \in X \text{ for all } y \in Y \}$

(2) the idealizer of Y mod X in L

$I = I_L(Y/X) = \{ x \in L : [Y,x] \leqslant Y \}$

(3) the quasicentralizer of Y mod X in L

$K = K_L(Y/X) = \{ x \in L : [y,x] \in \langle y \rangle + X \text{ for all } y \in Y \}$

If $X = 0$ we write C,I,K as $C_L(Y), I_L(Y), K_L(Y)$ respectively. If also $Y = \langle y \rangle$ we write $C_L(y)$ for $C_L(Y)$. Clearly $C \leqslant K \leqslant I$ and if $y \in Y$ and $k \in K$ then there exists $n \in Z$ such that $[y,k] = ny \bmod X$. Hence if $X \lhd Y$, k^* is a power derivation of Y/X.

Lemma 1.5

In the above notation $\langle [K,I] \rangle \leqslant C$

Proof

W.l.o.g we assume $X = 0$. Let $y \in Y$, $k \in K$ and $h \in I$; now $[y,[k,h]] = [h,y,k] + [y,k,h]$ and $[y,k] = ny$; but $[y,h] = y'$ for some $y' \in Y$ since $h \in I$, and so $[y',k] = my'$ for some $m \in Z$. Thus k^* is a power derivation of $\langle y,y' \rangle$ so by corollary 1.2.1 we may set $m = n$. It therefore follows that $[y,[k,h]] = -ny' + ny' = 0$, i.e. $\langle [K,I] \rangle \leqslant C$.

It also immediately follows that C and K are ideals of I.

Quasi-ideals

In connection with power derivations and quasicentralizers, we define a subring X of a Lie ring L to be a quasi-ideal of L, written X qi L, if for every $Y \leqslant L$, $\langle X,Y \rangle = X + Y$, or equivalently $[Y,X] \leqslant X + Y$. (Clearly ideals are quasi-ideals.) Then if $Y = \langle y \rangle$ and $x \in X$, it follows that $[y,x] = ny \bmod X$ for some $n \in Z$. Thus X qi L if and only if $X \leqslant K_L(L^*/X^*)$ (c.f. $X \lhd L$ if and only if $X \leqslant C_L(L^*/X^*)$)

Lie algebras

Let R be a commutative ring and L a (right) R-module (in the usual sense) then L is a Lie algebra over R if there is defined on L a bilinear multiplication $(x,y) \to [x,y]$ such that for all $x,y,z \in L$:

(1) $[x,x] = 0$

(2) $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$

(see [11] p351). Thus Lie rings are Lie algebras over Z. We define subalgebras, derivations, power derivations etc as for Lie rings; thus $\mathrm{Der}(L) \leqslant \mathrm{End}_R(L)$, the R-endomorphisms of L and if d is a power derivation of L then $xd = rx$ for some

$r \in R$ since $< x > = Rx$. If R is an integral domain we have the following generalization of lemma 1.1:

Lemma 1.6

If d is a power derivation of L and there exists $y \in L$ such that $ry = 0$ implies that $r = 0$ then d acts universally on L.

Proof

Let $x \in L$ then there exist $r,s,t \in R$ such that $xd = rx$, $yd = sy$, and $(x+y)d = t(x+y)$. Then $(s-t)sy = (s-t)yd = (t-r)xd = (t-r)rx = r(t-r)x = r(s-t)y = (s-t)ry$. So $(s-t)(s-r)y = 0$, therefore either $s = t$ or $s = r$. W.l.o.g. we assume $s = t$ then $(t-r)x = 0$ and $sx = tx = rx$. Therefore d acts universally on L.

In particular if R is a field all powerderivations are universal.

We define Lie algebra classes as we did for Lie rings. Thus if R is a field, $\underline{\underline{C}}$ is the class of one-dimensional Lie algebras and $\underline{\underline{T}}_n$ the class of Lie algebras of dimension $\leq n$. In the following chapters we shall always assume that L is a Lie ring, unless otherwise specified, but the results obtained will in general be applicable to Lie algebras over any commutative ring. An exception to this is chapter 6 where we must take into account the torsion structure of Lie rings.

CHAPTER 2

Series

A series of a Lie ring L is a set $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$...(*) of pairs of subgroups of L, where Σ is a linearly ordered set such that:

(i) $V_\sigma^* \leqslant \Lambda_\sigma^*$ for all $\sigma \in \Sigma$

(ii). $\Lambda_\sigma^* \leqslant V_\tau^*$ if $\sigma < \tau$

(iii) $L \setminus \{0\} = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \setminus V_\sigma)$

It follows that $V_\sigma = \bigcup_{\tau < \sigma} \Lambda_\tau$ and $\Lambda_\sigma = \bigcap_{\sigma < \tau} V_\tau$

The $\Lambda_\sigma \setminus V_\sigma$ are called the <u>layers</u> of the series (*) and (iii) implies that each $0 \neq x \in L$ lies in precisely one layer of the series (*).

If $V_\sigma \lhd \Lambda_\sigma$ for all $\sigma \in \Sigma$ the Λ_σ/V_σ are called the <u>factors</u> of the series (*). If also $\Lambda_\sigma/V_\sigma \in \underline{\underline{X}}$, where $\underline{\underline{X}}$ is some class for all $\sigma \in \Sigma$ then (*) is called an $\underline{\underline{X}}$-series of L and we write $L \in \hat{E}\underline{\underline{X}}$.

If $V_\sigma, \Lambda_\sigma \lhd L$ for all $\sigma \in L$ then (*) is called an <u>invariant</u> series of L.

If V_σ, Λ_σ ch L for all $\sigma \in L$ then (*) is called a <u>characteristic</u> series of L.

Ascending Series

If Σ is well ordered then Σ has the order type of an ordinal ρ, and we may take Σ to be the set of all ordinals $\sigma < \rho$. Then $\Lambda_\sigma = V_{\sigma+1}$ if $\sigma < \rho$ and if we define $V_\rho = L$ then the Λ_σ become superfluous and we may write(*) as an ascending series (V_σ : $\sigma \leqslant \rho$). We shall call ρ the length of the ascending series.

Descending Series

If Σ is inversely well ordered then Σ has the order type ρ^* for some ordinal ρ. Then if we relabel the terms of the series with the ordinals $\sigma < \rho$, replacing condition (ii) by (ii)* $\Lambda_\sigma \leqslant V_\tau$ if $\tau < \sigma$, it follows that $V_\sigma = \Lambda_{\sigma+1}$ if $\sigma < \rho$ and if we define $\Lambda_\rho = \{0\}$ the V_σ become superfluous and we may write (*) as a descending series (Λ_σ : $\sigma \leqslant \rho$).

Finite Series

If $|\Sigma| = n$, where $n \in Z$ we may write (*) as a finite series (V_i : $0 \leqslant i \leqslant n$)...(1). Suppose (W_j : $0 \leqslant j \leqslant m$).. (2) is another finite series of L, then:

(a) (2) is a refinement of (1) if for each $i \leqslant n$ there exists $j \leqslant m$ such that $V_i = W_j$

(b) (1) and (2) are isomorphic if there exists an injective map $\theta : \{1,\ldots,n\} \to \{1,\ldots,m\}$ such that $V_i/V_{i-1} \cong W_{\theta(i)}/W_{\theta(i)-1}$.

If L is an A-module for some Lie ring A and the Λ_σ, V_σ of (1) are A-submodules of L we call the series (1) A-invariant.

For finite series we have the following well-known result of Schreier which we state without proof (see [15])

Proposition 2.1

If L is an A-module and $(V_i : 0 \leqslant i \leqslant n)$ and $(W_j : 0 \leqslant j \leqslant m)$ are A-invariant series of L then they have A-invariant isomorphic refinements.

Paralyzers and Stabilizers of Series

Let L be a Lie ring and $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma) \ldots (*)$ a series of L and $D = \mathrm{Der}(L)$. We make the following definitions:

(1) $P = \bigcap_{\sigma \in \Sigma} K_D(\Lambda_\sigma/V_\sigma)$ is the paralyzer of $(*)$.

(2) $S = \bigcap_{\sigma \in \Sigma} C_D(\Lambda_\sigma/V_\sigma)$ is the stabilizer of $(*)$.

Suppose L is an A-module for some Lie ring A then:

(3) A paralyzes (stabilizes) $(*)$ if $A/C_A(L)$ is contained in the paralyzer (stabilizer) of $(*)$.

(4) A faithfully paralyzes (stabilizes) $(*)$ if A paralyzes (stabilizes) $(*)$ and $C_A(L) = 0$.

(5) A universally paralyzes $(*)$ if each $a \in A$ acts as a universal power derivation on every factor Λ_σ/V_σ of $(*)$.

(6) If ρ is some ordinal then A ρ-paralyzes (stabilizes) L if there exists an ascending series of L of length ρ paralyzed (stabilized) by A.

Note that if A paralyzes the series (*) then A also paralyzes the series $(\Lambda_\sigma^*, V_\sigma^* : \sigma \in \Sigma)$ of L^*. Hence if we wish to determine the Lie structure of A (or P or S) we may disregard the Lie structure of L i.e. assume L is an abelian Lie ring and D is the set of all endomorphisms of L.

Quasicentral, Central and Cyclic Series

Now L is an L-module under the adjoint action $yx^* = [y,$ where $x, y \in L$. If under this action L paralyzes (*) we shall call (*) a quasicentral series of L and write $L \in \underline{\underline{Q}}_\Sigma$. If L stabilizes the series (*) then we shall call (*) a central series of L and write $L \in \underline{\underline{Z}}_\Sigma$. If Σ is well-ordered of ordinali ρ we write $L \in \underline{\underline{Q}}_\rho$ and $L \in \underline{\underline{Z}}_\rho$. We also define:

$$\underline{\underline{Q}} = \bigcup_\Sigma \underline{\underline{Q}}_\Sigma \qquad \text{and} \qquad \underline{\underline{Z}} = \bigcup_\Sigma \underline{\underline{Z}}_\Sigma$$

where the unions are taken over all linearly ordered set Σ. Al

$$\underline{\underline{Q}}^A = \bigcup_\rho \underline{\underline{Q}}_\rho \qquad \text{and} \qquad \underline{\underline{Z}}^A = \bigcup_\rho \underline{\underline{Z}}_\rho$$

where the unions are taken over all ordinals. For finite ordin

we let $\quad \underline{\underline{P}} = \bigcup_{n<\omega} \underline{\underline{Q}}_n \qquad \text{and} \qquad \underline{\underline{N}} = \bigcup_{n<\omega} \underline{\underline{Z}}_n$

Then by analogy with group theory we call $\underline{\underline{Z}}^A$ the class of hypercentral Lie rings, $\underline{\underline{P}}$ the class of parasoluble Lie rings and $\underline{\underline{N}}$ the class of nilpotent Lie rings.

If $(V_\sigma : \sigma \leq \rho)$ is a quasicentral $\underline{\underline{C}}$-series of L then we shall say L is hypercyclic. Since a cyclic invariant series is a fortiori quasicentral we may equivalently say that L has an ascending invariant cyclic series. If ρ is finite we write $L \in \underline{\underline{S}}_\rho$ and define $\underline{\underline{S}} = \bigcup_{n<\omega} \underline{\underline{S}}_n$. This is the class of <u>supersoluble</u> Lie rings.

If $L \in \underline{\underline{S}}_n$ and $(V_i : 0 \leq i \leq n)$ is an invariant cyclic series of L then since $V_i/V_{i-1} \in \underline{\underline{C}}$ for $1 \leq i \leq n$ there exists $x_i \in L$ such that $V_i = \langle x_i \rangle + V_{i-1}$. In particular $L = V_n = \sum_{i=1}^{n} \langle x_i \rangle$, and so $L \in \underline{\underline{\Gamma}}_n$. Therefore $\underline{\underline{S}}_n \leq \underline{\underline{\Gamma}}_n$ and $\underline{\underline{S}} \leq \underline{\underline{\Gamma}}$.

All the classes $\underline{\underline{X}}$ defined above are <u>S-closed</u> i.e. $\underline{\underline{X}} = S\underline{\underline{X}}$ for if $K \leq L \in \underline{\underline{X}}$ and $(\Lambda_\sigma, V_\sigma \ ; \sigma \in \Sigma)$ is a quasicentral (central, invariant cyclic) series of L then clearly $(\Lambda_\sigma \cap K, V_\sigma \cap K \ : \sigma \in \Sigma)$ is a quasicentral (central, invariant cyclic) series of K.

Also all the classes $\underline{\underline{X}}$ with ascending or finite quasicentral (central, invariant cyclic) series are Q-closed since if $I \lhd L \in \underline{\underline{X}}$ and $(V_\sigma : \sigma \leq \rho)$ is such a series of L then $((V_\sigma + I)/I \ ; \sigma \leq \rho)$ is a quasicentral (central, invariant cyclic) series of L/I.

<u>Lemma 2.2</u>

If $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ is a quasicentral series of L then $\Lambda_\sigma/V_\sigma \in \underline{\underline{A}}$ for all $\sigma \in \Sigma$.

Proof

Let $\sigma \in \Sigma$ then since L paralyzes the series the adjoint action of each $x \in \Lambda_\sigma$ is as an inner power derivation of Λ_σ/V_σ Hence by lemma 1.4 $\Lambda_\sigma/V_\sigma \in \underline{\underline{A}}$.

Corollary 2.2.1

$\underline{\underline{P}} \leqslant P\underline{\underline{A}}$ i.e. parasoluble Lie rings are soluble.

Proof

If $L \in \underline{\underline{P}}$, L has a finite quasicentral series $(V_i : 0 \leqslant i \leqslant$ By lemma 2.2 this is an $\underline{\underline{A}}$-series and so $L \in P\underline{\underline{A}}$.

Lemma 2.2.2

$\underline{\underline{Q}}^A$ is the class of hypercyclic Lie rings.

Proof

If L is hypercyclic then clearly $L \in \underline{\underline{Q}}^A$. Conversely suppose that L has an ascending quasicentral series and X is any factor of this series. We may well-order the set of generators of X such that $X = < x_\sigma : \sigma < \rho >$ for some suitable ordinal ρ. Then if we set $X_\tau = < x_\sigma : \sigma < \tau >$ then $(X_\tau : \tau \leqslant \rho >$ is an ascending $\underline{\underline{C}}$-series of X which is L-invariant.

Now more generally let $K \lhd H \leqslant L$ and suppose that H/K has an ascending series paralyzed (stabilized) by L, then we say that H/K is an L-hypercyclic (L-hypercentral) factor of L.

An L-hypercyclic (L-hypercentral) series of L is one in which all the factors are L-hypercyclic (L-hypercentral). We define the class $\underline{\underline{Q}}^{AD}$ to be the class of Lie rings with a descending I hypercyclic series, and $\underline{\underline{Z}}^{AD}$ the class of Lie rings with a descending L-hypercentral series.

Note that if L has any quasicentral series then the factors of this series are L-hypercyclic factors of L.

<u>The Upper Central Series</u>

We define the <u>centre</u> of L, $Z(L) = \{ x \in L : [L,x] = 0\}$. Then we set $Z_1(L) = Z(L)$ and inductively define $Z_{\sigma+1}(L)$ by $Z_{\sigma+1}(L)/Z_\sigma(L) = Z(L/Z_\sigma(L))$ and $Z_\lambda(L) = \bigcup_{\sigma<\lambda} Z_\sigma(L)$ if λ is a limit ordinal. Then $Z_\sigma(L)$ ch L for all σ. If $L \in \underline{\underline{Z}}^A$ then $Z_\rho(L) = L$ for some ρ and the series $(Z_\sigma(L) \ : \sigma \leq \rho)$ is called the upper central series of L.

<u>The Lower Central Series</u>

If we write L^2 for $[L,L]$ then L^2 is the smallest ideal I of L such that $L/I \in \underline{\underline{A}}$. We set $L^1 = L$ and inductively define $L^{\sigma+1} = [L^\sigma, L]$ and $L^\lambda = \bigcap_{\sigma<\lambda} L^\sigma$ if λ is a limit ordinal. Then L cl for all σ. If L has a descending central series then $L^\rho = 0$ for some ρ and the series is called the lower central series of L.

Engel Lie Rings

L is an Engel Lie ring, written $L \in \underline{\underline{E}}$ if and only if for each ordered pair of elements $x,y \in L$ there exists $m = m(x,y) \in Z^+$ such that $[x,_m y] = 0$.

Lemma 2.3

If H/K is a quasicentral factor of L where $L \in \underline{\underline{E}}$, then H/K is an L-hypercentral factor of L. In particular if H/K is torsion-free or generated by elements of order p (a prime) then H/K is central in L.

Proof

Since K is necessarily an ideal of L and $Q\underline{\underline{E}} = \underline{\underline{E}}$ we may assume that $K = \{0\}$. Let $x \in H$ and $y \in L$ then $[x,y] = nx$ for some $n \in Z$. Therefore it follows (by induction) that $[x,_k y] = n^k x$ for all $k > 0$. Now since $L \in \underline{\underline{E}}$ there exists $m \in Z^+$ such that $[x,_m y] = 0$, and so $n^m x = 0$.

If $|x| = \infty$ then $n^m = 0$, so $n = 0$ and $[x,y] = 0$. If $|x| = p$ for some prime p then $p|n^m$ so $p|n$ and therefore again $[x,y] = 0$.

We now define $H_1 = < x \in H : |x| = \infty$ or a prime $p >$ and for $n > 1$ $H_n = < H_{n-1}, x \in H : |x| = p^n$ for some $p >$. Then $H = \bigcup_{n=1}^{\infty} H_n$ so if we set $H_\omega = H$ and $H_o = 0$ then $(H_i : 0 \leq i \leq \omega)$ is an ascending series of H stabilized by L.

Therefore H is L-hypercentral. Furthermore if H is torsion-free or generated by elements of order p then $H = H_1$ and so $[H,L] = 0$ i.e. H is central in L.

Note that if the torsion ideal of H has finite exponent then $H = H_n$ for some $n > 0$. We may therefore state:

Corollary 2.3.1

Parasoluble Engel Lie rings whose torsion ideals have finite exponent are nilpotent.

Another immediate corollary is:

Corollary 2.3.2

If H/K is an L-hypercyclic factor of the Engel Lie ring L then H/K is an L-hypercentral factor of L.

In particular $\underline{\underline{Q}}^A \cap \underline{\underline{E}} \leqslant \underline{\underline{Z}}^A$

For torsion-free Lie rings we may be more precise and state:

Corollary 2.3.3

If L is torsion-free and $L \in \underline{\underline{Q}}_\Sigma \cap \underline{\underline{E}}$ then $L \in \underline{\underline{Z}}_\Sigma$.

Note that this result also applies if L is an Engel Lie algebra over any field. For a field of characteristic 0 this is immediate and although a field of characteristic p has Z-torsion in the sense that $pL = 0$ if L is over such a field, lemma 5.5 still shows that quasicentral factors are central and so corollary 5.5.3 still holds.

Lemma 2.4

If $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ ...(*) is a series of L paralyzed by A and B is the subring of A which stabilizes (*) then $B \triangleleft A$ and $A/B \in \underline{\underline{A}}$. If also $|\Sigma| = n$ and A universally paralyzes (*) then $A/B \in \underline{\underline{\Gamma}}_n$.

Proof

Let $\sigma \in \Sigma$, $x \in \Lambda_\sigma$, $a \in A$ and $B \in B$, then $xb \in V_\sigma$ and $xa \in \Lambda_\sigma$ so clearly $x[a,b] = x(ab - ba) \in V_\sigma$ and $B \triangleleft A$. Now let $C_\sigma = C_A(\Lambda_\sigma/V_\sigma)$ then $A/C_\sigma \in \underline{\underline{A}}$ and $A/\bigcap_{\sigma\in\Sigma} C = A/B \in R\underline{\underline{A}} = \underline{\underline{A}}$. If also A universally paralyzes (*) then $A/C_\sigma \in \underline{\underline{C}}$ for all $\sigma \in \Sigma$ If $|\Sigma| = n$, let $\Sigma = \{ 1,\ldots,n \}$ then $A/B = A/\bigcap_{i=1}^{n} C_i$ which is isomorphic to a subring of $\coprod_{i=1}^{n} A/C_i \in \underline{\underline{\Gamma}}_n$. Since clearly $S\underline{\underline{\Gamma}}_n = \underline{\underline{\Gamma}}_n$ we deduce that $A/B \in \underline{\underline{\Gamma}}_n$.

Corollary 2.4.1

$$\underline{\underline{Q}}_\Sigma \leqslant \underline{\underline{Z}}_\Sigma \underline{\underline{A}}$$

Proof

Suppose $L \in \underline{\underline{Q}}_\Sigma$ and $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ is a quasicentral series of L. By lemma 2.4 L^2 stabilizes this series so $(\Lambda_\sigma \cap L^2, V_\sigma \cap L^2 : \sigma \in \Sigma)$ is a central series of L^2. Therefore $L^2 \in \underline{\underline{Z}}_\Sigma$ and $L \in \underline{\underline{Z}}_\Sigma \underline{\underline{A}}$.

Lemma 2.5

If L paralyzes the ascending series $(V_\sigma : \sigma \leqslant \rho)$... (*) of L where L is torsion-free then there exists an ascending series $(W_\sigma : \sigma \leqslant \rho)$ of L universally paralyzed by A.

Proof

Let $W_\sigma = \{ x \in L :$ there exists $k \in Z^*$ such that $kx \in V_\sigma \}$ for $\sigma \leqslant \rho$. Then $V_\sigma \leqslant W_\sigma$ and if λ is a limit ordinal and $x \in W_\lambda$ then $kx \in V_\lambda$ for some $k \in Z^*$. Since $V_\lambda = \bigcup_{\sigma<\lambda} V_\sigma$ it follows that $kx \in V_\sigma$ for some ordinal $\sigma < \lambda$ and so $x \in W_\sigma$. Therefore $W_\lambda = \bigcup_{\sigma<\lambda} W_\sigma$ and $(W_\sigma : \sigma \leqslant \rho)$ ...(2) is an ascending series of L. Furthermore each factor $W_{\sigma+1}/W_\sigma$ of (2) is torsion-free for suppose not then there exists $y \in W_{\sigma+1} \setminus W_\sigma$ such that $ny \in W$ for some $n \in Z^*$; this implies that $kny \in V_\sigma$ for some $k \in Z^*$ and therefore $y \in W_\sigma$ - a contradiction.

Now suppose $x \in W_\sigma$ and $x \neq 0$ so we may assume σ is not a limit ordinal. There exists $k \in Z^*$ such that $kx \in V_\sigma$, so if $a \in A$ then $kxa \equiv kmx \bmod V_{\sigma-1}$ since A paralyzes the series (1). Therefore $k(xa - mx) \in V_{\sigma-1}$ i.e. $xa - mx \in W_{\sigma-1}$. Hence $xa \equiv mx \bmod W_{\sigma-1}$ and so A paralyzes the series (2). Since each factor of (2) is torsion-free it follows from lemma 1.1 that A universally paralyzes (2).

Proposition 2.6

The paralyzer of any series lies in the class $\hat{E}\underline{\underline{A}}$.

Proof

Let A and B be the paralyzer and stabilizer respectively of the series $(\Lambda_\sigma, V_\sigma \ : \ \sigma \in \Sigma)$ of L. By lemma 2.2 $A/B \in \underline{\underline{A}}$ and we now show that B has an $\underline{\underline{A}}$-series using a method similar to that of the proof of lemma 4 of [6]. We well order a set of generators of L and suppose that $L = \langle x_\tau : \tau < \rho \rangle$ and set $L_\mu = \langle x_\tau : \tau < \mu \rangle$. then $L_\rho = L$, $L_0 = \{0\}$. Now let $C_\mu = C_B(L_\mu)$ then $C_{\mu+1} \leqslant C_\mu$ for all $\mu < \rho$,$C_0 = B$, $C \ = \{0\}$, and if λ is a limit ordinal then $C_\lambda = \bigcap_{\mu<\lambda} C_\mu$, and therefore $(C_\mu : \mu < \rho \ )$ is a descending series of B.

We now fix μ and write $x = x_\mu$, $X = C_{\mu+1}$, and $Y = C_\mu$ then $X = Y \cap C_B(x)$. We define:

$$Y_\sigma = \{ \ d \in Y \ : \ xd \in \Lambda_\sigma \ \}$$

$$X_\sigma = \{ \ d \in X \ : \ xd \in V_\sigma \ \}$$

Then if $d,e \in Y_\sigma$, by definition $xd, xe \in \Lambda_\sigma$ so $xed, xde \in V_\sigma$ Therefore $x[d,e] = xde - xed \in V_\sigma$ and so $[d,e] \in X_\sigma$. Since X_σ and Y_σ are clearly subgroups of B we deduce that $Y_\sigma / X_\sigma \in \underline{\underline{A}}$, Now writing $Y_\sigma = Y_{\sigma,\mu}$, $X_\sigma = X_{\sigma,\mu}$ we have $C_\mu \setminus C_{\mu+1} = \bigcup_{\sigma\in\Sigma} (Y_{\sigma,\mu} \setminus X_{\sigma,\mu})$. Hence $(Y_{\sigma,\mu}, X_{\sigma,\mu} : \sigma \in \Sigma, \ \mu < \rho \ )$ ordered by the relation $(\sigma_1, \mu_1) < (\sigma_2, \mu_2)$ if and only if $\mu_2 > \mu_1$ or $\mu_1 = \mu_2$

and $\sigma_1 < \sigma_2$, is an $\underline{\underline{A}}$-series of B. Hence B and therefore A lie in $\hat{E}\underline{\underline{A}}$.

If the series of L above is a descending series this proof also shows that A also has a descending $\underline{\underline{A}}$-series. However the following proposition gives a more precise result:

<u>Proposition 2.7</u>

If A is the paralyzer of the descending series $(\Lambda_\sigma : \sigma \leq \rho)$ of L then A has a descending $\underline{\underline{A}}$-series $(A_\sigma : \sigma \leq 1+\rho)$.

<u>Proof</u> (see [6] p12)

Let B be the stabilizer of the series and for $\sigma \leq \rho$ let $C_\sigma = C_B(L^*_\sigma/\Lambda^*_\sigma)$. Then $C_0 = B$, $C_\rho = \{0\}$ and if λ is a limit ordinal then $C_\lambda = \bigcap_{\sigma<\lambda} C_\sigma$. Therefore since $C_\sigma \lhd B$ for all $\sigma \leq \rho$ $(C_\sigma : \sigma \leq \rho)$ is a descending series of B.

Now if $\sigma < \rho$, then $<[\Lambda_\sigma, B]> \leq \Lambda_{\sigma+1}$ and $<[L, C_\sigma]> \leq \Lambda_\sigma$ so $<[L, C_\sigma, C_\sigma]> \leq \Lambda_{\sigma+1}$. By the Jacobi identity we have $<[C_\sigma, C_\sigma, L]> \leq <[L, C_\sigma, C_\sigma]>$ so $<[L, C^2_\sigma]> \leq \Lambda_{\sigma+1}$. Hence $C^2_\sigma \leq C_{\sigma+1}$ and therefore $C_{\sigma+1}/C_\sigma \in \underline{\underline{A}}$. Also $A/B \in \underline{\underline{A}}$ by lemma 2.2 so we complete the proof by defining $A_1 = B$ and $A_{1+\sigma} = C_\sigma$ for $1 \leq \sigma \leq \rho$. Note that if ρ is infinite then $1+\rho = \rho$.

We also have the following similar result for ascending series, the proof of which again follows that of [6] p 13.

Proposition 2.8

If A is the paralyzer of the ascending series $(V_\sigma : \sigma \leq \rho)$ of L then A has a descending $\underline{\underline{A}}$-series $(B_\sigma ; \sigma \leq 1+\rho)$.

Proof

Let B be the stabilizer of the series and for $\sigma \leq \rho$ let $C_\sigma = C_B(V_\sigma)$. Then $C_0 = B$, $C_\rho = \{0\}$ and if λ is a limit ordinal then $C_\lambda = \bigcap_{\sigma<\lambda} C_\sigma$. Also $C_\sigma \lhd B$ for all $\sigma \leq \rho$. so $(C_\sigma : \sigma \leq \rho)$ is an invariant descending series of B. Since $<[V_{\sigma+1}, B]> \leq V_\sigma$ and $[V_\sigma, C_\sigma] = 0$ if $\sigma < \rho$ it follows that $<[V_{\sigma+1}, C_\sigma, C_\sigma]> = 0$. Hence as in proposition 2.7 we deduce that $[V_{\sigma+1}, C_\sigma^2]> = 0$ and $C_\sigma^2 \leq C_{\sigma+1}$ so $C_{\sigma+1} / C_\sigma \in \underline{\underline{A}}$. Also $A/B \in \underline{\underline{A}}$ by lemma 2.2 so we now set $B_1 = B$ and $B_{1+\sigma} = C_\sigma$ for $\sigma \leq \rho$ to complete the proof.

Corollary 2.8.1

The paralyzer of a finite series is soluble.

Proof

If we set ρ to be finite in either proposition 2.7 or proposition 2.8 we deduce that $A \in P\underline{\underline{A}}$.

We conclude this chapter with the following proposition for invariant series. The proof is similar to that of lemma 5 of [6] p 12)

Proposition 2.9

If A paralyzes the invariant series $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ of L then $\langle [L,A]^A \rangle \in \underline{\underline{Z}}_\Sigma$

Proof

Let $P = L + A$ (see chapter 1). We let $K_\sigma = K_P(\Lambda_\sigma / V_\sigma)$ then $K_\sigma \lhd P$ for all $\sigma \in \Sigma$ by lemma 1.5 and $A \leqslant \bigcap_{\sigma \in \Sigma} K_\sigma$. Therefore $[L,A] \leqslant C_P(\Lambda_\sigma / V_\sigma) \lhd P$ again by lemma 1.5. Hence $H = \langle [L,A]^A \rangle = \langle [L,A]^P \rangle \lhd P$ and $H \leqslant C_P(\Lambda_\sigma / V_\sigma)$ for all $\sigma \in \Sigma$.

Now consider the series $(H \cap \Lambda_\sigma, H \cap V_\sigma : \sigma \in \Sigma)$ of H. Let $h \in H$ and $x \in H \cap \Lambda_\sigma$ then $[x,h] \in H \cap V_\sigma$ since $H \lhd P$. Therefore this is a central series of H and $H \in \underline{\underline{Z}}_\Sigma$.

CHAPTER 3

We consider here paralyzers of finite series of modules.

Theorem 3.1

If A is a Lie ring which universally faithfully paralyzes the finite series of A-modules $(V_i : 0 \leq i \leq m)$ ...(1) then $A \in \underline{\underline{Q}}_c$ where $c = (m^2-m+2)/2$.

Proof

Let $B = \bigcap_{i=1}^{n} C_A(V_i/V_{i-1})$ then B faithfully stabilizes the series (1). For $1 \leq k \leq m-1$ and $0 \leq i \leq k$ we define:

$C_k = B \cap C_A(V_k)$ and $D_{k,i} = B \cap C_A(V_k) \cap C_A(V_{k+1}/V_i)$. Then $C_{k+1} = D_{k,o} \leq D_{k,1} \leq \dots \leq D_{k,k} = C_k$ ...(2), and $D_{k,i} \lhd A$. Note that $D_{m-1,0} = 0$ and $D_{1,1} = B$. We shall show that the series (2) is universally paralyzed by A (under the adjoint action)

Let $d \in D_{k,i}$ where $i > 0$ and suppose that for $1 \leq j \leq m$ that $x_j \in V_j$. Then $x_j d = 0$ if $j \leq k$, $x_{k+1}d = y_i$ for some $y_i \in V_i$ and $x_j d = y_{j-1}$ for some $y_{j-1} \in V_{j-1}$ if $j > k+1$. Let $a \in A$ then for all $x_j \in V_j$ $(j > 1)$ there exists $n_j \in Z$ such that $x_j a = n_j x_j + z_{j-1}$ for some $z_{j-1} \in V_{j-1}$, since A universally paralyzes the series (1). Therefore we have $x_{k+1}da = n_i y_i + w_{i-1}$ for some $w_{i-1} \in V_{i-1}$ and $x_{k+1}ad = n_{k+1}y_i$

So $[a,d] = ad - da \equiv (n_i - n_{k+1})d \bmod C_A(V_{k+1}/V_{i-1})$

Also $x_j[a,d] = 0$ if $j \leqslant k$ and $x_j[a,d] \in V_{j-1}$ if $j > k+1$ so $[a,d] = (n_i - n_{k+1})d \bmod B \cap C_A(V_k)$. Thus we deduce that $[a,d] = (n_i - n_{k+1})d \bmod D_{k,i-1}$ and the series (2) is paralyzed by A. Hence there exists a series of B of length $\sum_{k=1}^{m-1} k = m(m-1)/2$ paralyzed by A. Since $B \lhd A$, and $A/B \in \underline{\underline{A}}$ by lemma 2.4, $A \in \underline{\underline{Q}}_c$ where $c = m(m-1)/2 + 1 = (m^2 - m+2)/2$.

Note also that it follows that $L + A \in \underline{\underline{Q}}_e$ where $e = (m^2 - m + 2)/2 + m = (m^2+m+2)/2$.

Suppose L is a p-Lie ring and $(V_i : 0 \leq i \leq m)$ is a series of L faithfully paralyzed by A. Now L is also a $Z_{\bar{p}}$-module and if $\bar{r} \in Z_{\bar{p}}$ and $x \in L$, $|x| = p^n$ then $\bar{r}x = (r_0 + r_1 p + \ldots + r_{m-1}p^{m-1})x$. Hence the $Z_{\bar{p}}$-submodules of L are precisely the Z-submodules. Also if $a \in A$ we may define $\bar{r}a$ to be the map $\bar{r}a : x \rightarrow (\bar{r}x)a$. Hence we may equivalently say that the series above is faithfully paralyzed by A, where we regard both L and A as Lie algebras over $Z_{\bar{p}}$. Moreover we know from chapter 1 that the series is universally paralyzed in this sense. The proof of theorem 3.1 shows that A is parasoluble as a Lie algebra over $Z_{\bar{p}}$.

<u>Corollary 3.1.1</u> (c.f. Hall [4])

If B is a Lie ring which faithfully stabilizes the finite series of B-modules $(V_i : 0 \leq i \leq m)$ then $B \in \underline{\underline{Z}}_c$ where $c = m(m-1)/2$.

<u>Proof</u>

In the proof of theorem 3.1 put A = B and $n_i = 0$ for $1 \leq i \leq m$. However we can improve this to the analogue of Kaloujnine's result [9]: let $V = V_m$ then by the Jacobi identity and induction $[V, B^k] \leq [V,_k B]$ if $k > 0$. Since $[V,_m B] = 0$ we deduce that $[V, B^m] = 0$ and as $C_B(V) = 0$ it follows that $B^m = 0$ i.e. $B \in \underline{\underline{Z}}_{m-1}$. ~~Again this can be demonstrated by a matrix method, since we can represent the elements of B as upper-triangular matrices.~~

The following two lemmas are immediate consequences of theorem 3.1.

Lemma 3.2

If A faithfully universally paralyzes the ascending series $(V_n : n \leqslant \omega)$ of L then $A \in R\underline{\underline{P}}$.

Proof

Let $C_n = C_A(V_n)$ if $n \leqslant \omega$ then $C_n \lhd A$ and A/C_n faithfully universally paralyzes the finite series $(V_i : 0 \leqslant i \leqslant n)$. Hence by theorem 3.1 $A/C_n \in \underline{\underline{P}}$. But $\bigcap_{n=0}^{\infty} C_n = 0$ since $L = \bigcup_{n=0}^{\infty} V_n$ so $A \in R\underline{\underline{P}}$.

Similarly we can show that the stabilizer of the series lies in $R\underline{\underline{N}}$.(c.f. lemma 3 of [6]).

Lemma 3.3

If A faithfully universally paralyzes the descending series $\langle \Lambda_n : n \leqslant \omega)$ of L then $A \in R\underline{\underline{P}}$.

Proof

We may assume $L \in \underline{\underline{A}}$ so we let $C_n = C_A(L^*/\Lambda_n^*)$ for $n \leqslant \omega$ then $C_n \lhd A$ and A/C_n faithfully universely paralyzes the finite series $(\Lambda_{n-i}^*/\Lambda_n^* : 0 \leqslant i \leqslant n)$ of L^*/Λ_n^*. Hence $A/C_n \in \underline{\underline{P}}$ by theorem 3.1. But $\bigcap_{n=0}^{\infty} \Lambda_n = \Lambda_\omega = 0$ so $\bigcap_{n=0}^{\infty} A_n = 0$ and $A \in R\underline{\underline{P}}$.

Again we can show that the stabilizer of this series lies in $R\underline{\underline{N}}$ (c.f. lemma 3 of [6]).

The condition that A should <u>universally</u> paralyze the series in theorem 3.1 is necessary in general for A to be parasoluble is shown by the following example.

Example

Let $V = C \oplus X$, $V \in \underline{\underline{A}}$ where $C \cong Z_{p^\infty}$ and $X = \langle x \rangle \cong Z$ (as abelian groups), and A the paralyzer of the series $0 \lhd C \lhd C + X = V \ldots (*)$. We shall show that A is not even hypercyclic.

Every ordered triple $(n, y, \bar{p})$ where $n \in Z$, $y \in C$, and $\bar{p} \in Z_{\bar{p}}$ determines an endomorphism (and therefore a derivation) d of V given by: $\quad xd = nx + y$

$$\text{and} \qquad ad = \bar{p}a \quad \text{for all } a \in C,$$

The set of all such d clearly paralyzes the series (*) and furthermore every element of A has this form. Suppose A is parasoluble, then there exists a non-zero ideal E of A such that if $0 \neq e \in E$ then $[e, d] = ke$ for some $k \in Z$.

$$\text{Suppose} \quad xe = mx + z \quad \text{where } z \in C$$

$$\text{and} \qquad ae = \bar{q}a \quad \text{where } \bar{q} \in Z_{\bar{p}} \text{ for all } a \in C$$

Then $x[e, d] = mnx + my + \bar{p}z - mnx - nz - \bar{q}y = kmx + kz$.

Therefore $kmx = 0$ so either $m = 0$ or $k = 0$.

If $m = 0$ then $(\bar{p} - n - k)z = \bar{q}y$ so if $|z| = p^r$ then $p^r\bar{q}y = 0$. Since this is true for all $y \in C$, which has infinite exponent we deduce that $\bar{q} = 0$. Hence $(\bar{p} - n - k)z = 0$; but $\bar{p}$ and n are again arbitrary so $z = 0$ and e is the zero map. Therefore $k = 0$ and $(\bar{p} - n)z = (\bar{q} - m)y$. Again y is arbitrary so $(\bar{q} - m) = 0$ and $z = 0$. Thus $xe = mx$ and $ae = ma$. Furthermore $[e,d] = 0$ so $e \in Z(A)$ and $\langle e \rangle \lhd A$; also every element in $Z(A)$ is of a form similar to that of e, i.e. a universal power derivation of V.

We continue by shewing that if I is any ideal of A contained in $Z(A)$ and f is an element of A such that $I + \langle f \rangle \lhd A$, then $f \in Z(A)$. Since $A \notin \underline{\underline{A}}$ it then follows that A is not hypercyclic, and a fortiori not parasoluble.

Suppose $xf = jx + v$, where $j \in Z$ and $v \in C$, and $af = \bar{s}a$ for all $a \in A$, where $\bar{s} \in Z_{\bar{p}}$. Then it follows that $xfd = njx + jy + \bar{p}v$ and $xdf = jnx + nv + \bar{s}y$ and therefore $x[f,d] = (j - \bar{s})y + (\bar{p} - n)v = rjx + rv + tx$ for some $t \in Z$. Hence $(rj + t)x = 0$ so $rj = -t$, and $(\bar{p} - n - r)v = (j - \bar{s})y$; y is arbitrary so $(j - \bar{s}) = 0$ therefore $(\bar{p} - n - r)v = 0$ and so $v = 0$. Hence $xf = jx$ and $af = ja$, and again $f \in Z(A)$.

CHAPTER 4

In [6] it is shown that the stability group of an ascending series of a group lies in the class of groups corresponding to our $L\underline{\underline{R}}\underline{\underline{N}} \cap \underline{\underline{Z}}^{AD}$. In this chapter we shall obtain the analogue of this result for Lie rings as a corollary to a stronger result, namely that the paralyzer of an ascending series of Lie ring-modules lies in the class $L\underline{\underline{R}}\underline{\underline{S}} \cap \underline{\underline{Q}}^{AD}$.

Proposition 4.1

If A paralyzes the ascending series $(V_\sigma : \sigma \leqslant \rho)\ldots(1)$ of L and B is any $\underline{\underline{G}}$-subring of A then $\langle x^B \rangle \in \underline{\underline{T}}$ for each $x \in L$.

Proof

Let $s = (a_i)$ be any sequence of elements of A and $x \in L$. We define sequences $s_1 = (\sigma_i)$ of ordinals and $s_2 = (x_i)$ of elements of L by the following: we let σ_1 be the least ordinal for which $[x, a_1] \in \langle x \rangle + V_{\sigma_1}$ and we then choose $x_1 \in V_{\sigma_1}$ such that $[x, a_1] \in \langle x \rangle + \langle x_1 \rangle \ldots (2)$. Inductively we define σ_k for $k > 1$ to be the least ordinal for which $[x_{k-1}, a_k] \in \langle x_{k-1} \rangle + V_{\sigma_k}$ and choose $x_k \in V_{\sigma_k}$ such that $[x_{k-1}, a_k] \in \langle x_{k-1} \rangle + \langle x_k \rangle \ldots (3)$

Since A paralyzes the series (1) it is clear that:

(i) σ_k is not a limit ordinal

(ii) $\sigma_k < \sigma_{k-1}$ if $\sigma_k \neq 0$

hence (iii) $\sigma_k = 0$ for some k

Writing now more explicitly $\sigma_k = \sigma_k(s)$, clearly $\sigma_k(s)$ depends only on the first k elements of s...(4) Now $[x,a_1,\ldots,a_n]$ lies in the Lie ring $<x> + \{<x_{\sigma_k(s)}> ; 1 \leq k \leq n, s \in S\}$ (5) where S is the set of all sequences whose first n terms lie in $\{1, \ldots, n\}$. This we show by induction on n; if n = 1 it follows from (2) and the induction step is by (3).

Now suppose $B = < a_i : 1 \leq i \leq m >$ then:

$< x^B > = < x, \{ \Sigma[x,y_1,\ldots,y_k] ; k > 0 \} >$ ... (6)

and $y_j \in \{ a_i : 1 \leq i \leq m \}$ if $j \leq k$.

By (4) $| \{\sigma_k(s) : s \in S\} | \leq m^k$. Now $\sigma_1(s) < \rho$ for all $s \in S$ so $\tau_1 = \mathrm{Max}\{ \sigma_1(s) : s \in S \} < \rho$ Similarly if we define $\tau_k = \mathrm{Max}\{ \sigma_k(s) : s \in S \}$ for $k > 1$ then $\tau_k < \tau_{k-1}$ if $\tau_k \neq 0$ and so $\tau_N = 0$ for some $N \in Z$.

Hence by (5) and (6):

$< x^B > = < x > + \{ < x_{\sigma_k(s)} > : k < N \}$ and so $< x^B > \in \underline{\underline{\Gamma}}$

<u>Proposition 4.2</u>

$$\underline{\underline{G}} \cap \underline{\underline{Q}}^A \leq \underline{\underline{\Gamma}}$$

<u>Proof</u>

Suppose $L \in \underline{\underline{G}} \cap \underline{\underline{Q}}^A$ and $L = < u_1,\ldots,u_n >$ then we may write $L = \sum_{i=1}^{n} < u_i^L >$. By proposition 4.1 $< u_i^L > \in \underline{\underline{\Gamma}}$ for $1 \leq i \leq n$ and hence $L \in \underline{\underline{\Gamma}}$.

Corollary 4.2.1

$$\underline{\underline{G}} \cap \underline{\underline{Q}}^A \leq \underline{\underline{S}}$$

Proof

If $L \in \underline{\underline{G}} \cap \underline{\underline{Q}}^A$ then by proposition 4.2 $L \in \underline{\underline{\Gamma}}$ or equivalently $L^* \in \underline{\underline{G}}$. Hence L^* satisfies the maximal condition on subgroups so any ascending cyclic series of L has only finitely many distinct terms. Since L is hypercyclic we deduce that L has a finite invariant $\underline{\underline{C}}$-series i.e. L is supersoluble.

For finitely-generated parasoluble Lie-rings we have the following more precise result (c.f. [17] lemma 13.8).

Proposition 4.3

$$\underline{\underline{Q}}_n \cap \underline{\underline{G}}_m \leq \underline{\underline{\Gamma}}_{f(n,m)} \quad \text{where } f(n,m) = m(m^n-1)/(m-1).$$

Proof

Let $L \in \underline{\underline{Q}}_n \cap \underline{\underline{G}}_m$ and $L = < y_1, \ldots, y_m >$ and suppose that $(V_i : 0 \leq i \leq n)$ is a quasicentral series of L. Let $x = y_1$ then $(< x^L > \cap V_i : 0 \leq i \leq n >$ is a quasicentral series of $< x^L >$. Now we set $a_i = y_i$ for $1 \leq i \leq m$ and define τ_k as in the proof of proposition 4.1; then it is clear that $\tau_1 \leq n-1$, $\tau_k \leq n-k$ if $k < n$ and $\tau_k = 0$ if $k \geq n$. Therefore $< x^L > = < x > + \{ \Sigma[x, a_{i_1}, \ldots, a_{i_k}] : k < n \}$ (see (5) of proposition 4.1). Hence $< x^L >^*$ is generated by $1 + m + \ldots + m^{n-1} \doteq (m^n - 1)/(m-1)$ elements.

Therefore $L^* = \sum_{i=1}^{m} < y_i^L >$ is generated by $m(m^n-1)/(m-1)$ elements, i.e. $L \in \underline{\underline{\Gamma}}_{f(n,m)}$.

Note that it also immediately follows that $L \in \underline{\underline{S}}_{f(n,m)}$.

We now prove the first of the main results of this chapter.

Theorem 4.4

If A faithfully paralyzes the ascending series $(V_\sigma : \sigma \leqslant \rho)$ of L then $A \in L R \underline{\underline{S}}$.

Proof

Let B be a $\underline{\underline{G}}$-subring of A and suppose $B = < a_1, \ldots, a_m >$. If $a \in B$ then there exists $x \in L$ such that $xa \neq 0$. We let $X = < x^B >$ and $C = C_B(X)$ then $C \lhd B$, $a \notin C$ and B/C faithfully paralyzes the ascending series $(V_\sigma \cap X : \sigma \leqslant \rho)$ of X. Also since $B \in \underline{\underline{G}}$, $< x^B > \in \underline{\underline{\Gamma}}$ by proposition 4.1. Therefore the series $(V_\sigma \cap X ; \sigma \leqslant \rho)$ has only finitely many distinct terms (c.f. corollary 4.2.1) and since each factor of this series lies in $\underline{\underline{\Gamma}}$, by corollary 1.2.1 B/C faithfully universally paralyzes a finite series of L. Hence B/C is parasoluble by theorem 3.1. Since $B/C \in Q\underline{\underline{G}} = \underline{\underline{G}}$, proposition 4.3 implies that $B/C \in \underline{\underline{S}}$. Hence $B \in R\underline{\underline{S}}$ and $A \in LR\underline{\underline{S}}$.

Corollary 4.4.1 (c.f. theorem A2 of [6])

If A faithfully stabilizes the ascending series $(V_\sigma : \sigma \leqslant \rho)$ then $A \in LR\underline{\underline{N}}$.

Proof

Using the notation of theorem 4.4 we have that B/C faithfully stablizes the series $(V_\sigma : \sigma \leq \rho)$ which we may assume to be finite, so by corollary 3.1.1, $B/C \in \mathfrak{N}$. Therefore $B \in R\mathfrak{N}$ and $A \in LR\mathfrak{N}$.

Countably Recognizable Classes

We define the closure operation L_ω by the following: given any class $\mathfrak{X}$, $L \in L_\omega\mathfrak{X}$ if and only if every countable subring of L lies in $S\mathfrak{X}$. Clearly $L\mathfrak{X} \leq L_\omega\mathfrak{X}$; $\mathfrak{X}$ is said to be countably recognizable if $\mathfrak{X} = L_\omega\mathfrak{X}$.

Proposition 4.5

$$L_\omega\mathfrak{Q}^A = \mathfrak{Q}^A$$

Proof

We consider a Lie ring $L \notin \mathfrak{Q}^A$ and shew it has a countable subring not in $\mathfrak{Q}^A$. We construct countable subrings L_i of L inductively as follows: let $0 \neq x \in L$ and $< x > = L_0$. Suppose we have constructed $L_0, \ldots, L_i$; since $L \notin \mathfrak{Q}^A$ there exists, for each $y \in L_i$, an element $u_y \in L$ such that $[y, u_y] \notin \, <y>$. Let $L_{i+1} = \, < L_i, u_y : 0 \neq y \in L_i >$, then L_{i+1} is countable. Let $M = \bigcup_{i=0}^{\infty} L_i$ then M is a countable subring of L, and clearly contains no non-zero cyclic ideals.

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We now prove the second of the main results of this chapter.

Theorem 4.6

If A faithfully paralyzes the ascending series $(V_\sigma : \sigma \leqslant \rho)$ of L then $A \in \underline{\underline{Q}}^{AD}$.

Proof

We assume, without loss of generality, that $V_{\sigma+1}/V_\sigma \in \underline{\underline{C}}$ for all $\sigma < \rho$. Then we choose x_σ for all non-limit ordinals $\sigma \leqslant \rho$ such that $V_\sigma = \langle x_\tau : \tau \leqslant \sigma \rangle$ then if λ is a limit ordinal, $V_\lambda = \langle x_\tau : \tau < \lambda \rangle$. We may assume that A is the paralyzer of $(V_\sigma : \sigma \leqslant \rho)$ since $\underline{\underline{Q}}^{AD} = S\underline{\underline{Q}}^{AD}$, and define B to be the stabilizer. Then $B \lhd A$ and $A/B \in \underline{\underline{A}}$ by lemma 2.4. For $0 \leqslant \sigma \leqslant \rho$ let $C_\sigma = C_B(V_\sigma)$ then, as in the proof of proposition 2.8 $(C_\sigma : \sigma \leqslant \rho)$ is a descending $\underline{\underline{A}}$-series of B. For each $\sigma \leqslant \rho$ we now define for $0 \leqslant \tau \leqslant \sigma$ $D_{\sigma,\tau} = C_\sigma \cap C_B(V_{\sigma+1}/V_\tau)$ Then $D_{\sigma,\sigma} = C_\sigma$, $D_{\sigma,0} = C_{\sigma+1}$ and $D_{\sigma,\tau} \leqslant D_{\sigma,\tau+1}$ if $\tau < \sigma$. Also if λ is a limit ordinal then $D_{\sigma,\lambda} = \bigcup_{\tau<\lambda} D_{\sigma,\tau}$ since if $d \in D_{\sigma,\lambda}$ then $x_{\sigma+1}d \in V = \bigcup_{\tau<\lambda} V_\tau$ so $x_{\sigma+1}d \in V_\tau$ for some $\tau < \lambda$ and $d \in D_{\sigma,\tau}$. Also $D_{\sigma,\tau} \lhd A$ since $D_{\sigma,\tau} = B \cap C_A(V_\sigma) \cap C_A(V_{\sigma+1}/V_\tau)$ so $(D_{\sigma,\tau}/C_{\sigma+1} : \tau \leqslant \sigma)$ is an ascending series of $C_\sigma/C_{\sigma+1}$. Now suppose that $d \in D_{\sigma,\tau+1} \setminus D_{\sigma,\tau}$ for some $\tau < \sigma$ then $x_\mu d = 0$ if $\mu \leqslant \sigma$, $x_\mu d \in V_{\mu-1}$ if $\mu > \sigma+1$ and $x_{\sigma+1}d \equiv nx_{\tau+1} \bmod V_\tau$ for some $n \in Z$.

Let e be the endomorphism of L given by $x_\mu e = 0$ if $\mu \neq \sigma+1$ and $x_{\sigma+1}e = x_{\sigma+1}$. Then e is a derivation of L^*, and since we can assume that $L \in \underline{\underline{A}}$, $e \in D_{\sigma,\tau+1}$. Furthermore $x_\mu(d - ne) = 0$ if $\mu < \sigma$, $x_\mu(d - ne) \in V_{\mu-1}$ if $\mu > \sigma+1$ and $x_{\sigma+1}(d - ne) \in V_\tau$ i.e. $d = ne \bmod D_{\sigma,\tau}$ so $D_{\sigma,\tau+1}/D_{\sigma,\tau} = < e + D_{\sigma,\tau} >$ and $D_{\sigma,\tau+1}/D_{\sigma,\tau} \in \underline{\underline{C}}$. Hence $C_\sigma/C_{\sigma+1}$ is an A-hypercyclic factor of A. Since $A/B \in \underline{\underline{A}}$ and $(C_\sigma : \sigma \leq \rho)$ is a descending series of B, we deduce that $A \in \underline{\underline{Q}}^{AD}$.

Note that the example in chapter 3 shows that A need not lie in $\underline{\underline{Q}}^A$.

<u>Corollary 4.6.1</u>(c.f. theorem A1 of [6]).

If B faithfully stabilizes the ascending series $(V_\sigma : \sigma \leq \rho)$ of L then $B \in \underline{\underline{Z}}^{AD}$

<u>Proof</u>

We use the notation of theorem 4.6 and let $d \in D_{\sigma,\tau+1} \setminus D_{\sigma,\tau}$. Now let $b \in B$ then $x_\mu b \in V_{\mu-1}$ for all non-limit ordinals $\mu \leq \rho$. Therefore $x_\mu db = x_\mu bd = x_\mu[d,b] = 0$ if $\mu < \sigma+1$, and $x_\mu db \equiv x_\mu bd \equiv x_\mu[d,b] \equiv 0 \bmod V_\nu$ for some $\nu < \mu$ if $\mu < \sigma+1$. $x_{\sigma+1}db \in V_\tau$ and $x_{\sigma+1}bd = 0$ so $x_{\sigma+1}[d,b] \in V_\tau$ and it follows that $[d,b] \in D_{\sigma,\tau}$ and $[D_{\sigma,\tau+1}, B] \leq D_{\sigma,\tau}$. Hence $C_\sigma/C_{\sigma+1}$ is a hypercentral factor of B and so since $(C_\sigma : \sigma \leq \rho)$ is a descending series of B, $B \in \underline{\underline{Z}}^{AD}$.

<u>Corollary 4.6.2</u>

If A faithfully paralyzes the finite cyclic series $(V_i : 0 \leq i \leq n)$ of L then $A \in \underline{\underline{S}}_c$ where $c = (n^2+n)/2$.

<u>Proof</u>

We use the notation of the proof of theorem 4.6 with $\rho = n$. Then B has an A-invariant cyclic series of length $\sum_{r=0}^{n} r = n(n-1)/2$. By lemma 2.4, $A/B \in \underline{\underline{A}} \cap \underline{\underline{\Gamma}}_n$ so A has an invariant cyclic series of length $n(n-1)/2 + n = c$. Hence $A \in \underline{\underline{S}}_c$.

Note that we could have deduced the fact that $A \in \underline{\underline{S}}$ from theorem 3.1 and proposition 4.3, since $L \in \underline{\underline{\Gamma}}$ and so Der(L) and therefore $A \in \underline{\underline{\Gamma}}$

CHAPTER 5

In this chapter we consider some join properties of hypercyclic ideals of a Lie ring.

We introduce here some new classes of Lie rings:

(i) $L \in \underline{\underline{Q}}^*_\rho$ if L has a characteristic ascending quasicentral series of length ρ.

(ii) $L \in \underline{\underline{S}}^*_n$ if L has a characteristic cyclic series of length n.

Thus, for example, $\underline{\underline{Z}}_\rho \leq \underline{\underline{Q}}^*_\rho \leq \underline{\underline{Q}}_\rho$.

The following assertions are immediate:

(1) If $I \lhd L$ and $I \in \underline{\underline{S}}^*_n$, $L/I \in \underline{\underline{S}}_m$ then $L \in \underline{\underline{S}}_{n+m}$.

(2) If $L \cong H \oplus K$ where $H \in \underline{\underline{Q}}_\rho$ and $K \in \underline{\underline{Q}}_\tau$ then $L \in \underline{\underline{Q}}_{(\rho+\tau)}$.

<u>Proposition 5.1</u> (c.f. Lemma 2.1 of [17])

Suppose $H \lhd H + K = L$, $H \in \underline{\underline{Q}}^*_{\rho_1}$, $K/(K \cap H) \in \underline{\underline{Q}}_{\rho_2}$ and K (with adjoint action) ρ_3-paralyzes H. Then $L \in \underline{\underline{Q}}_{(\rho_1\rho_3+\rho_2)}$

<u>Proof</u>

Let a characteristic quasicentral series of H be $(C_\sigma : \sigma \leq \rho_1)$ and let $(B_\tau : \tau \leq \rho_3)$ be a series of H which is paralyzed by K. If λ is a limit ordinal then $B_\lambda \cap C_{\sigma+1} = \bigcup_{\mu<\lambda}(B_\mu \cap C_{\sigma+1})$, so $((B_\tau \cap C_{\sigma+1})C_\sigma/C_\sigma : \tau \leq \rho_3)\ldots(*)$ is an ascending series of $(H \cap C_{\sigma+1})/C_\sigma$.

Let $x \in B_\tau \cap C_{\sigma+1}$, $x \neq 0$ then we may assume that $\tau \neq 0$ and τ is not a limit ordinal. Let $k \in K$, $h \in H$, then $[x,k] \equiv nx \mod B_{\tau-1} \cap C_{\sigma+1}$ for some $n \in Z$ since $C_\sigma \lhd L$, and also $[x,h] \equiv mx \mod C_\sigma$ for some $m \in Z$.

Hence (*) is a series of $C_{\sigma+1}/C_\sigma$ paralyzed by $H + K$ and therefore L. Thus H is $\rho_1\rho_3$-paralyzed by L.

But $(H+K)/H \cong K/(K\cap H) \in \underline{\underline{Q}}_{\rho_2}$ so $L \in \underline{\underline{Q}}_{\rho_1\rho_3+\rho_2}$

<u>Proposition 5.2</u>

If $L = H + K$ where $H,K \lhd L$, $H \in \underline{\underline{Q}}^*_\rho$ and $K \in \underline{\underline{Q}}_{\rho'}$ then $L \in \underline{\underline{Q}}_{\rho'\rho+\rho+\rho'}$

<u>Proof</u>

Let $(H_\sigma : \sigma \leqslant \rho)$ be a characteristic quasicentral series of H and $(K_\tau : \tau \leqslant \rho')$ a quasicentral series of K. Therefore $(K \cap H_\sigma : \sigma \leqslant \rho)$ is an ascending series of $K \cap H$ and since if λ is a limit ordinal $K_\lambda \cap H_{\sigma+1} = \bigcup_{\tau<\lambda}(K_\tau \cap H_{\sigma+1})$ we also have that $((K_\tau \cap H_{\sigma+1})(K \cap H_\sigma)/(K \cap H_\sigma)\ ;\ \tau \leqslant \rho')\ldots(*)$ is an ascending series of $(K \cap H_{\sigma+1})/(K \cap H_\sigma)$.

Suppose $x \in K_\tau \cap H_{\sigma+1}$ then if $x \neq 0$ we may assume that τ is not a limit ordinal or zero. Let $h \in H$ and $k \in K$. Then $[x,h] = nx \mod H_\sigma \cap K$ for some $n \in Z$ since $K \lhd L$ and $[x,k] = mx \mod H_{\sigma+1} \cap K_{\tau-1}$ for some $m \in Z$ since $H_{\sigma+1}$ ch H and $H \lhd L$ so $H_{\sigma+1} \lhd L$.

Hence $H + K = L$ paralyzes the series (*), and therefore L $\rho'\rho$-paralyzes $K \cap H$.

Also $(H + K)/(H \cap K) \cong H/(H \cap K) \oplus K/(H \cap K) \in \underline{\underline{Q}}_{\rho+\rho'}$

It therefore follows that $L \in \underline{\underline{Q}}_{\rho'\rho+\rho+\rho'}$

Corollary 5.2.1

(1) The join of a hypercentral ideal and a hypercyclic ideal is hypercyclic.

(2) The join of a nilpotent ideal and a parasoluble ideal is parasoluble (c.f. proposition 1.7 of [8]).

Proof

(1) is immediate from proposition 5.2 since $\underline{\underline{Z}}^{A} \leqslant \bigcup_{\rho} \underline{\underline{Q}}^{*}_{\rho}$

For (2) we let ρ and ρ' be finite in proposition 5.2 and use (1).

In the next proposition we give necessary and sufficient conditions for the join of two hypercyclic ideals of a Lie ring to be hypercyclic. We need first the following lemma.

Lemma 5.3

Suppose $Y \lhd L$ and $x \in C_L(Y^2)$. Then if $n > 1$ and $\sigma \in S_n$ (the symmetric group on n letters) and $y_1, \ldots, y_n \in Y$

$[x, y_1, \ldots, y_n] = [x, y_{\sigma(1)}, \ldots y_{\sigma(n)}]$.

Proof

Let $C = C_L(Y^2)$ then $C \lhd Y$ by lemma 1.5, since $Y^2 \lhd L$.

Let $z \in C$ then if $0 \leq i,j \leq n$, $zy_i^*y_j^* = zy_j^*y_i^*$ because $[z,y_i,y_j] = [z,y_j,y_i] - [y_i,y_j,z] = [z,y_j,y_i]$ since $[y_i,y_j]$ lies in L^2. It now follows that $xy_1^*\dots y_n^* = xy_{\sigma(1)}^*\dots y_{\sigma(n)}^*$ since $xy^* \in C$ for all $y \in C$ and the fact that any $\sigma \in S_n$ is generated by transpositions.

<u>Proposition 5.4</u>

Let $L = H + K$ where $H,K \lhd L$ and $H,K \in \underline{\underline{Q}}^A$. Then $L \in \underline{\underline{Q}}^A$ if and only if $L^2 \in \underline{\underline{Z}}^A$.

<u>Proof</u>

If $L \in \underline{\underline{Q}}_\rho$ then $L^2 \in \underline{\underline{Z}}_\rho$ by corollary 2.4.1. Conversely suppose that $L^2 \in \underline{\underline{Z}}_{\rho_1}$, $H \in \underline{\underline{Q}}_{\rho_2}$, and $K \in \underline{\underline{Q}}_{\rho_3}$. Then by corollary 5.2.1, $H + L^2 = A$ and $K + L^2$ are hypercyclic ideals of L and $L = A + B$. More specifically $A \in \underline{\underline{Q}}_{\rho_4}$ and $B \in \underline{\underline{Q}}_{\rho_5}$ where $\rho_4 = \rho_2\rho_1 + \rho_1 + \rho_2$ and $\rho_5 = \rho_3\rho_1 + \rho_1 + \rho_3$.

Suppose $(A_\sigma : \sigma \leq \rho_4)$ and $(B_\tau : \tau \leq \rho_5)$ are quasicentral series of A and B respectively, and $Z_1 = Z_1(L^2)$ then $Z_1 \neq 0$ since $L^2 \in \underline{\underline{Z}}^A$. Let $N = \langle (Z \cap B_\tau)^L \rangle$ for $0 \leq \tau \leq \rho_5$ then since if λ is a limit $N_\lambda = \bigcup_{\tau<\lambda} N_\tau$, $(N_\tau : \tau \leq \rho_5)\dots(*)$ is an ascending series of Z_1. Let $u \in N_\tau$, $u \neq 0$, then we may assume that τ is not a limit and $u = \sum_{i=1}^m u_i$ where each u_i has the form $u_i = [x_i,y_1,\dots,y_k]$ where $x_i \in Z_1 \cap B_\tau$ and $y_j \in L$ for $1 \leq j \leq k$.

Suppose $b \in B$ then $[u_i, b] = [u_i, y_1, \dots, y_k, b]$ which by lemma 5.3 is equal to $[u_i, b, y_1, \dots, y_k]$. Since b^* is a power derivation of $B_\tau / B_{\tau-1}$, by corollary 1.2.1 we can choose $n \in Z$ such that $[x_i, b] \equiv nx_i \bmod B_{\tau-1}$ for all $1 \leq i \leq m$. Then $[u_i, b] = [nx_i + w_i, y_1, \dots, y_k]$ for some $w_i \in B_{\tau-1} \cap Z_1$ since $Z_1 \lhd L$. We can write this as $[u_i, b] = n[x_i, y_1, \dots, y_k] + [w_i, y_1, \dots, y_k]$ so that $[u_i, b] \equiv n[x_i, y_1, \dots, y_k] \bmod \langle (B_{\tau-1} \cap Z_1)^L \rangle$. Therefore $[u, b] \equiv nu \bmod N_{\tau-1}$ and the series (*) is paralyzed by B.

Now consider the series $(N_\tau(N_{\tau+1} \cap A_\sigma)/N_\tau : \sigma \leq \rho_4)$ of $N_{\tau+1}/N_\tau$. This is paralyzed by both A and B and therefore by $A + B = L$. Hence Z_1 is $\rho_4\rho_5$-paralyzed by L. We can now apply the same hypotheses to L/Z_1 and argue by transfinite induction to show that $Z_{\sigma+1}(L^2)/Z_\sigma(L^2)$ is $\rho_4\rho_5$-paralyzed by L for all $0 \leq \sigma \leq \rho_1$, noting that $Z_\lambda(L^2) = \bigcup_{\sigma < \lambda} Z_\sigma(L^2)$ for all limit ordinals $\lambda \leq \rho_1$. It therefore follows that L^2 is $\rho_4\rho_5\rho_1$-paralyzed by L.

Since $L/L^2 \in \underline{\underline{A}}$ we deduce that $L \in \underline{\underline{Q}}_{\rho_4\rho_5\rho_1 + 1}$

<u>Corollary 5.4.1</u>(C.f. [8] theorem 1.9 p42)

If $L = H + K$ where $H, K \lhd L$ and $H, K \in \underline{\underline{P}}$ then $L \in \underline{\underline{P}}$ if and only if $L^2 \in \underline{\underline{N}}$.

Proof

If L is parasoluble then L^2 is nilpotent by corollary 2.4.1. Conversely if $L^2 \in \underline{\underline{N}}$ then in the notation of the proof of proposition 5.4 ρ_1, ρ_2, ρ_3 are finite and so also is $\rho_4 \rho_5 \rho_6 + 1$ and therefore $L \in \underline{\underline{P}}$.

Corollary 5.4.2

If $L = H + K$ where $H, K \lhd L$ and $H, K \in \underline{\underline{S}}$ then $L \in \underline{\underline{S}}$ if and only if $L^2 \in \underline{\underline{N}}$.

Proof

If $L \in \underline{\underline{S}}$ then $L \in \underline{\underline{P}}$ so $L^2 \in \underline{\underline{N}}$ by corollary 2.4.1. Conversely if $L^2 \in \underline{\underline{N}}$ and $H, K \in \underline{\underline{S}}$ then $H, K \in \underline{\underline{P}} \cap \underline{\underline{G}}$ so by corollary 5.4.1 $L \in \underline{\underline{P}} \cap \underline{\underline{G}}$. Now by corollary 4.2.1. $L \in \underline{\underline{S}}$.

It is shown in [7] lemma 7 that the join of all $L\underline{\underline{N}}$ ideals of a Lie algebra over a field lies in $L\underline{\underline{N}}$. This is also true for Lie rings. We can now generalize this to give necessary and sufficient conditions for the join of all $L\underline{\underline{S}}$ ideals (i.e. the $L\underline{\underline{P}}$ ideals, by corollary 4.2.1) of a Lie ring to lie in $L\underline{\underline{S}}$.

Proposition 5.5

If $\{ H_\sigma : \sigma \in \Sigma \}$ is a set of $L\underline{\underline{S}}$ ideals of L then $J = \langle H_\sigma ; \sigma \in \Sigma \rangle \in L\underline{\underline{S}}$ if and only if $J^2 \in L\underline{\underline{N}}$.

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Proof

Suppose $J^2 \in L\underline{\underline{N}}$ and C is a $\underline{\underline{G}}$-subring of J. Then there exist $\sigma_1,\ldots,\sigma_s \in \Sigma$ such that $C \leqslant \sum_{i=1}^{s} H_{\sigma_i}$. Without loss of generality we suppose $s = 2$ and $C = < h_1,\ldots,h_m,k_1,\ldots,k_n >$ where $h_i \in H_{\sigma_1} = H$ for $1 \leqslant i \leqslant m$ and $k_j \in H_{\sigma_2} = K$ for $1 \leqslant j \leqslant n$. Let $A = < h_1,\ldots,h_m >$ and $B = < k_1,\ldots,k_n >$ then $X = < A^C > = < S, h_i : 1 \leqslant i \leqslant m >$ where S is the set $\{ \Sigma[h_i,k_{j_1},\ldots,k_{j_r}] : 1 \leqslant i \leqslant m\ ,\ 1 \leqslant j \leqslant n,\ \ r > 0 \}$. But $S \leqslant < k_j,\ [h_i,k_j] : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n >$ which is a $\underline{\underline{G}}$-subring of K and therefore lies in $\underline{\underline{S}} \leqslant \underline{\underline{T}}$. Hence $X \in \underline{\underline{G}}$ and $X \leqslant H$ and so $X \in \underline{\underline{S}}$. Similarly if we define $Y = < B^C >$ then $Y \in \underline{\underline{S}}$. Since $C^2 \in \underline{\underline{G}}$ and $C^2 \leqslant J^2$, $C^2 \in \underline{\underline{N}}$ so $C = X + Y$ lies in $\underline{\underline{S}}$ by corollary 5.4.2. Hence $J \in L\underline{\underline{S}}$.

Conversely suppose that $J \in L\underline{\underline{S}}$ then $J \in L(\underline{\underline{NA}})$ by corollary 5.4.2. If W is a $\underline{\underline{G}}$-subring of L^2 there exist $n \in Z$, $u_i,v_i \in L$ for $1 \leqslant i \leqslant n$ such that $W = < [u_i,v_i] : 1 \leqslant i \leqslant n >$. If $V = < u_i,v_i : 1 \leqslant i \leqslant n >$ then $V \in \underline{\underline{G}}$ so $V \in \underline{\underline{NA}}$ and $W \leqslant V^2 \in \underline{\underline{N}}$. Therefore $W \in S\underline{\underline{N}} = \underline{\underline{N}}$ and $J^2 \in L\underline{\underline{N}}$.

CHAPTER 6

We define the class $\underline{\underline{T}}$ as follows: $L \in \underline{\underline{T}}$ if and only if $H \lhd K \lhd L$ implies $H \lhd L$. In this chapter we shall look at the class $P\underline{\underline{A}} \cap \underline{\underline{T}}$; the corresponding classes of groups and Lie algebras over fields have been studied by Robinson [12] and Stewart [16]. The connection with quasicentral series is given by the following proposition:

Proposition 6.1

$L \in P\underline{\underline{A}} \cap \underline{\underline{T}}$ if and only if L is soluble and every finite abelian series of L is a quasicentral series.

Proof

Suppose $L \in P\underline{\underline{A}} \cap \underline{\underline{T}}$ and $(H_i : 0 \leqslant i \leqslant n)(*)$ is an $\underline{\underline{A}}$-series of L (of which there must exist at least one.) Since $L \in \underline{\underline{T}}$ $H_i \lhd L$ for $0 \leqslant i \leqslant n$. Let $h \in H_i$ for some $i \neq 0$ then $< h + H_{i-1} > \lhd H_i/H_{i-1} \lhd L/H_{i-1} \in Q\underline{\underline{T}} = \underline{\underline{T}}$. Therefore we have $< h + H_{i-1} > \lhd L/H_{i-1}$, so if $x \in L$ then $[h,x] \equiv mh \bmod H_{i-1}$ for some $m \in Z$. Hence $(*)$ is a quasicentral series.

Conversely suppose L is soluble and every finite abelian series of L is quasicentral. Let H si L then by proposition 2.1 there exists a finite abelian series of L with H as one of its terms. Since this is a quasicentral series, $H \lhd L$. Therefore $L \in \underline{\underline{T}}$.

To continue we need the following two lemmas which are analogues of results of Robinson [12].

Lemma 6.2

If $L \in \underline{\underline{T}}$ and $C = C_L(L^2)$ then C is the unique maximal nilpotent ideal of L and is abelian.

Proof

Let $N \lhd L$ and $N \in \underline{\underline{N}}$. As in group theory every subring of a nilpotent Lie ring is a subideal, hence if $x \in N$, $\langle x \rangle$ si N so $\langle x \rangle$ si $L \in \underline{\underline{T}}$ and therefore $\langle x \rangle \lhd L$. Thus the adjoint action of L on x is by power derivations and so $L/C_L(x) \in \underline{\underline{A}}$, therefore $L^2 \leqslant C_L(x)$. Hence $x \in C_L(L^2) = C$ and so $N \leqslant C$. Also $C^3 \leqslant [L^2,C] = 0$ so $C \in \underline{\underline{N}}$. Therefore evèry subring of C is an ideal of L and so of C, so by lemma 1.4, $C \in \underline{\underline{A}}$.

Note that lemma 1.4 also immediately implies that $\underline{\underline{N}} \cap \underline{\underline{T}} = \underline{\underline{A}}$.

Lemma 6.3

$$P\underline{\underline{A}} \cap \underline{\underline{T}} \leqslant \underline{\underline{A}}^2$$

Proof

Suppose not; then since $\underline{\underline{T}} = Q\underline{\underline{T}}$ there exists $L \in \underline{\underline{A}}^3 \cap \underline{\underline{T}}$, $L \neq 0$, so if $H = [L^2,L^2]$ then $H \in \underline{\underline{A}}$ and $H \neq 0$. If $h \in H$ then $\langle h \rangle \lhd H \lhd L$ so $\langle h \rangle \lhd L$. Now as in lemma 6.2,

$L/C_L(h) \in \underline{\underline{A}}$ so $L^2 \leqslant C_L(h)$ and $[H,L^2] = 0$. Therefore $L^2 \in \underline{\underline{N}}$ and if $x \in L^2$, $\langle x \rangle$ si L^2 so $\langle x \rangle \lhd L$ and $L^2 \leqslant C_L(x)$, i.e. $[L^2,L^2] = 0$ - a contradiction.

Corollary 6.3.1

$$P\underline{\underline{A}} \cap \underline{\underline{T}} \leqslant \underline{\underline{Q}}_2$$

Corollary 6.3.2

If $L \in P\underline{\underline{A}} \cap \underline{\underline{T}}$ and $C = C_L(L^2)$ then $C_L(C) = C$ i.e. C is self-centralizing.

Proof

If $K = C_L(C)$ then since $L^2 \in \underline{\underline{A}} \leqslant \underline{\underline{N}}$, $L^2 \leqslant C$ by lemma 6.2 and so $[L^2,K] = 0$ and $K \leqslant C$. Also by lemma 6.2 $C \in \underline{\underline{A}}$ so $C \leqslant K$ and therefore $C = K$.

Hence if $L \in P\underline{\underline{A}} \cap \underline{\underline{T}}$ then L can be written in the form $C + X$ where $C = C_L(L^2) \in \underline{\underline{A}}$ and is self-centralizing. If $x \in C$ then $\langle x \rangle \lhd C \lhd L$ so $\langle x \rangle \lhd L$ and for each $y \in L$ there exists $n \in Z$ such that $[x,y] = nx$. Trivially $\underline{\underline{A}} \leqslant \underline{\underline{T}}$ so as in [12] we classify the <u>non-abelian</u> members of $P\underline{\underline{A}} \cap \underline{\underline{T}}$ into three types depending on the structure of C.

(1) C contains a torsion-free element.

Then if $y \in L$ there exists $n \in Z$ such that $[x,y] = nx$ for all $x \in C$. If $my = 0$ mod C then $[x,my] = mnx = 0$ so L/C is torsion-free. Also if $[x,z] = nx$ for all $x \in C$ and some $z \in L$ then $y - z \in C_L(C) = C$, so $L/C \cong Z$.

(2) C is periodic and if $[C_p,L] \neq 0$ then $\exp(C_p) < \infty$. Denote the set of such primes by P, then if $y \in L$ there exist integers $0 < n_p < \exp(C_p)$ for each $p \in P$ such that if $x_p \in C_p$ then $[x_p,y] = n_p x_p$. If $|P| = \infty$ and $my \in C$ for some $m \in Z$ then $mn_p = 0 \bmod \exp(C_p)$ for all $p \in P$ so $m = 0$ and L/C is not periodic. If $C \in \underline{\underline{G}}$ then C has finite exponent e and there exists $n \in Z$ such that $[x,y] = nx$ for all $x \in C$. Since $ex \in C$, $L/C \cong Z_d$ where $d|e$.

(3) C is periodic and for some prime p, $\exp(C_p) = \infty$ and $[C_p,L] \neq 0$. Hence there exists $y \in L$ such that $[C_p,y] \neq 0$ and a uniquely defined $r \in Z_{\bar{p}}$ such that $[x,y] = rx$ for all $x \in C_p$. Hence L/C is not periodic.

In lemmas 6.4.1 to 6.4.6 we shall assume that L is a non-abelian $P\underline{\underline{A}} \cap \underline{\underline{T}}$ Lie ring of types (1) or (2), and $C = C_L(L^2)$. Also in lemmas 6.4.1 to 6.4.3 we shall assume C is either in $\underline{\underline{G}}$ or not periodic. Then in either case $L = C + \langle u \rangle$ where $[a,u] = na$ for all $a \in C$ for some $n \in Z$. We choose $x \in L \setminus C$, then x has the form $x = a + mu$ where $a \in C$, $m \in Z$. Then if $e = \exp(C)$, $0 < m < e$.

Lemma 6.4.1

$$< u^L > \ = nC + < u >$$

Proof

$$< u^L > \ = \ <u> + \{\Sigma [u, y_1, \ldots, y_r] : y_i \in L, \ i \leqslant r, \ r > 0\}$$

Suppose $y_i = a_i + m_i u$ where $a_i \in C$ and $m_i \in Z$, then $[u, y_1] = -na_1$, $[u, y_1, y_2] = m_2 na_1$ and $[u, y_1, \ldots, y_r] = m_r \ldots m_1 na_1$

Therefore $< u^L > \ = nC + <u>$.

Lemma 6.4.2

$$< x^L > \ = \ <x> + <na> + mnC$$

Proof

Let $I = \ <x> + <na> + mnC$ then I is an ideal of L since $<na> \ \lhd L$, $mnC \lhd L$ and $[x, a_i + m_i u] = m_i na + nma_i$ which lies in $<na> + mnC$. Furthermore I is the smallest ideal of L containing x since $[x,u] = na$ and $[x,C] = mnC$. Therefore $< x^L > \ = I$.

Lemma 6.4.3

$$<x> + <na> + mnC = \ <x> + <mn^2 a> + m^2 n^2 C$$

Proof

$$< x^I > \ = \ <x> + \{\Sigma [u, z_1, \ldots, z_r] ; z_i \in I, \ i \leqslant r \}$$

where $I = \ < x^L >$. Also $[x, na] = mn^2 a$ and $[x, mnC] = m^2 n^2 C$. Since $L \in \underline{\underline{T}}$, $< x^I > \ = \ < x^L >$ and the result follows.

Lemma 6.4.4

If C has a torsion-free element, then nC is divisible (i.e. $mnC = nC$ for all $0 \neq m \in \mathbb{Z}$).

Proof

Since x is torsion-free, $\langle x \rangle \cap C = \{0\}$ so by lemma 6.4.3, $\langle na \rangle + mnC = \langle mn^2a \rangle + m^2n^2C \ldots (*)$
If $m = 1$ and $a = 0$, $nC = n^2C$. If $m \neq 1$ and $a = 0$, $mnC = m^2nC$ hence $mn^2a \in mnC = m^2nC = m^2n^2C$ so by $(*)$ $na \in mnC$. This is true for all $a \in C$ so $nC \leqslant mnC$ for all $m \neq 0$ and so nC is divisible.

Lemma 6.4.5

If C is torsion-free then C is divisible.

Proof

Suppose not then there exists $a \in C$ and $0 \neq m \in Z$ such that there does not exist $b \in C$ with $mb = a$. But by lemma 6.4.4 nC is divisible so there exists $c \in C$ such that $na = mnC$, so $n(a - mc) = 0$ and $a = mc$ - a contradiction.

Hence in this case C as an abelian group is isomorphic to a direct sum of copies of Q the rational numbers. (See [14])

Lemma 6.4.6

If L is of type (2) and $\exp(C) = p^s$, then $s = 1$.

Proof

By lemma 6.4.3:

$$\langle x\rangle + \langle na\rangle + mnC = \langle x\rangle + \langle mn^2a\rangle + m^2n^2C.$$

If $m = 1$ and $a = 0$, $\langle u\rangle + nC = \langle u\rangle + n^2C$ therefore $[nC,u] = [n^2C,u]$ and so $n^2C = n^3C$ ...(1)

If $m \neq 1$ and $a = 0$, $\langle mu\rangle + mnC = \langle mu\rangle + m^2n^2C$, therefore $[mnC,u] = [m^2n^2C,u]$ and so $mn^2C = m^2n^3C$. Hence by (1) $mn^2C = m^2n^2C$ ...(2)

Suppose $s > 1$, then setting $m = p$ in (2) we have $p^2n^2C = pn^2C$ so that $pn^2C = 0$. If $n^2C = 0$, then by lemmas 6.4.2 and 6.4.3, $\langle y^L\rangle = \langle y\rangle$ for all $y \in L$, so by lemma 1.4 $L \in \underline{\underline{A}}$ - a contradiction. Therefore n^2C has exponent p and $n^2C = p^{s-1}C$, so $n^2 = p^{s-1}k$ where $p \nmid k$. Hence s is odd and may be written $s = 2r+1$ for some $r \in Z^*$. Then $n = p^rj$ where $p \nmid j$. By (1) $p^{2r}C = p^{3r}C = 0$ if $r > 0$. Therefore $r = 0$ i.e. $s = 1$ - a contradiction. Thus $\exp(C) = p$.

Lemma 6.5

If L has the form $C + \langle u\rangle$ where $C \in \underline{\underline{A}}$ and $[c,u] = nc$ for all $c \in C$ and either (i) nC is divisible or (ii) $\exp(C) = p$, then $L \in \underline{\underline{T}}$.

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Proof

Let $J \lhd\lhd I = \langle J^L \rangle \lhd L$. If $J \leqslant C$ then $[J,L] = nJ$ and $J \lhd L$ so suppose there exists $a \in C$ such that $a+mu \in J \setminus C$ for some $m \in Z^*$. Then $[a+mu,C] = mnC \leqslant I$ so J contains $[a+mu,mnC] = m^2n^2C$.

(i) If nC is divisible, $m^2n^2C = nC = L^2$ so $J \lhd L$

(ii) If $\exp(C) = p$ then $p \nmid m$ and either $nC = 0$ in which case $L \in \underline{\underline{A}} \leqslant \underline{\underline{T}}$, or $mnC = C = m^2n^2C$ and again $L^2 \leqslant J$ so $J \lhd L$.

Note that this result shows that p-Lie rings in $P\underline{\underline{A}} \cap \underline{\underline{T}}$ have exponent p^2. (or are abelian).

Lemma 6.6

If L has the form $C + \langle u \rangle$ where $C \in \underline{\underline{A}}$ and

(i) $[C_p,u] = 0$ unless $p \in \{p_1,\dots p_t\}$

(ii) $\exp(C_p) = p$ if $p \in \{p_1,\dots,p_t\}$

(iii) $[a,u] = na$ if $a \in C_{p_i}$ for some $1 \leqslant i \leqslant t$

(iv) $|u| = p_1 \dots p_t d$ where $p_i \nmid d$ if $1 \leqslant i \leqslant t$

then $L \in \underline{\underline{T}}$.

Proof

Let $J \lhd I = \langle J^L \rangle \lhd L$ and $K = \coprod_{i=1}^{t} C_{p_i}$. If $J \leqslant C$ then $[J,L] \leqslant J$ and $J \lhd L$ so suppose $x \in J \setminus C$. If we write $x = a + b + mu$ where $a \in K$, $b \in C \setminus K$ or $b = 0$ and

$1 \leqslant m \leqslant |u|$. Now $I_x = \langle x^L \rangle = \langle x \rangle + \langle na \rangle + mK$ and $\langle x^{I_x} \rangle = \langle x \rangle + \langle mn^2 a \rangle + m^2 K$. Also $\langle na \rangle = \langle a \rangle$ since $\exp(C_{p_i}) = p_i$ for $1 \leqslant i \leqslant t$. Let $e = \text{h.c.f.}(|a|, m)$ then if $\text{h.c.f.}(|u|, m) = f$, $e | f$ and $\text{h.c.f.}(|u|/f, e) = 1$ since $|u|/d$ is a product of different primes. Therefore $\langle a \rangle \in \langle a + mu \rangle + \langle ea \rangle = \langle a + mu \rangle + \langle mn^2 a \rangle$ and so $\langle x^{I_x} \rangle = \langle x^L \rangle$. Also $\langle y^L \rangle = \langle y \rangle$ if $y \in C$ so $I = \sum_{x \in J} \langle x^L \rangle = \sum_{x \in J} \langle x^{I_x} \rangle \leqslant \langle J^I \rangle = J$ so $J \lhd L$.

The next result is the analogue of theorem 3.3.1 of [12].

<u>Proposition 6.7</u>

$$\underline{\underline{G}} \cap P\underline{\underline{A}} \cap \underline{\underline{T}} \leqslant \underline{\underline{A}} \cup \underline{\underline{F}}$$

<u>Proof</u>

Suppose $L \in \underline{\underline{G}} \cap P\underline{\underline{A}} \cap \underline{\underline{T}}$ $\underline{\underline{A}}$ and $C = C_L(L^2)$ and C is not periodic. Then by lemma 6.4.6 L has the form $C + \langle u \rangle$ where $[a, u] = na$ ...(1) for all $a \in C$. Let the generators of L be $a_i + m_i u$ for $1 \leqslant i \leqslant s$ then $C = \sum_{i=1}^{s} \langle a_i \rangle$ since the action of u on a_i is by (1). Thus $C \in \underline{\underline{G}} \cap \underline{\underline{A}} \leqslant \underline{\underline{T}}$ and so $nC \in \underline{\underline{T}}$; but nC is divisible by lemma 6.4.6 so $nC = 0$ which implies that $L \in \underline{\underline{A}}$ - a contradiction. Therefore C is periodic and abelian, so $C \in \underline{\underline{F}}$.

Let $X = <[a_i + m_i u, a_j + m_j u] \; ; \; 1 \leqslant i,j \leqslant s >$ then $X \lhd L$ so $X = L^2$ and $L^2 \in \mathfrak{F}$ since $X \leqslant C$. Also L/C is isomorphic to a subring of $\mathrm{Der}(L^2) \in \mathfrak{F}$ so $L/C \in \mathfrak{F}$ and $L \in \mathfrak{F}^2 = \mathfrak{F}$.

Our final result in this chapter concerns $\bar{\mathfrak{T}}$, defined as the largest S-closed subclass of $\mathfrak{T}$.

<u>Proposition 6.8</u>

The non-abelian soluble $\bar{\mathfrak{T}}$-Lie rings are non-abelian $P\mathfrak{A} \cap \mathfrak{T}$ Lie rings of type (2).

<u>Proof</u>

Let L be a non-abelian $P\mathfrak{A} \cap \bar{\mathfrak{T}}$ Lie ring and $C = C_L(L^2)$. Suppose C contains an element b of infinite order, then by lemma 7.4.4 $L = C + <u>$ where $[c,u] = nc$ if $c \in C$ and nC is divisible. Then $<nb> + <u> \in P\mathfrak{A} \cap \mathfrak{T}$ but $<nb>$ is not divisible unless $n = 0$ which implies $L \in \mathfrak{A}$ - a contradiction.

Hence C is periodic. Suppose $\exp(C_p) = \infty$ and $[C_p, L] \neq 0$. Then there exists $y \in L \;\; C$ with $|y| = \infty$, and $c \in C_p$ such that $[c,y] \neq 0$. Then $<x,c> \in \mathfrak{G} \cap P\mathfrak{A} \cap \mathfrak{T}$ so by proposition 6.7 $<x,c> \in \mathfrak{A} \cup \mathfrak{F}$ - a contradiction.

Therefore L is a non-abelian $P\mathfrak{A} \cap \mathfrak{T}$ Lie ring of type (2).

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CHAPTER 7

In this chapter we will show that $L\underline{\underline{Q}}_n \leqslant \underline{\underline{Q}}_{2n}$ and $L\underline{\underline{Q}}_n \cap \underline{\underline{X}} \leqslant \underline{\underline{Q}}_n$ if $\underline{\underline{X}}$ is either the class of periodic Lie rings or the class of torsion-free Lie rings.

Suppose $L \in L\underline{\underline{Q}}_n$, then $L \in L(\underline{\underline{Z}}_n\underline{\underline{A}}) = \underline{\underline{Z}}_n\underline{\underline{A}}$ by corollary 2.4.1. Let $K = L^2$ and $0 = K^{n+1} \leqslant K^n \leqslant \ldots \leqslant K^1 = K$ be the lower central series of K; then since K^j ch $K \lhd L$, $K^j \lhd L$ for $0 \leqslant j \leqslant n$. Let $A = L/K$ then $A \in \underline{\underline{A}}$ and each factor $M = K^j/K^{j+1}$ of the lower central series of K is an A-module under the natural action:

$$(x + K^{j+1})(a + L^2) = [x,a] + K^{j+1} \qquad \ldots(1)$$

since $[K^j,L^2] \leqslant K^{j+1}$ and where $x \in K^j$, $a \in L$.

Now let $M = K^n$, $x_1,\ldots,x_r \in M$, $b_1,\ldots,b_s \in L$ then $X = < x_1,\ldots,x_r,b_1,\ldots,b_s > \in \underline{\underline{Q}}_n$. Suppose therefore that $(X_i : 0 \leqslant i \leqslant n)$ ... (2) is a quasicentral series of X. We shall now show that if $Y_i = < (M \cap X_i)^X>$ for $0 \leqslant i \leqslant n$ then $(Y_i : 0 \leqslant i \leqslant n)$ is a series of $Y = <(M \cap X)^X>$ paralyzed by X. Let $y \in Y_i$ for some $1 \leqslant i \leqslant n$ and $b \in X$ then $[y,b] = my + z$ where $z \in X_{i-1}$ and $m \in Z$, since $Y_i \leqslant X_i$ and (2) is a quasicentral series of X. But $[y,b], my \in Y_i$ so $z \in Y_i \cap X_{i-1} = Y_{i-1}$ (since Y_i and X_{i-1} are both ideals of X containing $M \cap X_{i-1}$). Now writing $b_j = a_j + w_j$ where

$a_j \in L \setminus L^2$ or $a_j = 0$ and $w_j \in L^2$ for $1 \leqslant j \leqslant s$, we have $[y,b_j] = [y,a_j] = ky \bmod Y_{i-1}$ for some $k \in Z$, and since $[m,L^2] = 0$. Hence under the action (1), $(X \cap M)^X$ is n-paralyzed by $B = (X + L^2)/L^2$... (3)

Since $K^j \lhd L$ for $1 \leqslant j \leqslant n$ and $L\underline{\underline{Q}}_n = QL\underline{\underline{Q}}_n$ we may say (3) is true when M is any factor K^j/K^{j+1} of the lower central series of L^2 and X is any $\underline{\underline{G}}$-subring of L/L^2. This motivates the following definitions:

If M is an A-module then A <u>locally n-paralyzes</u> (locally n-stabilizes) M if for every $\underline{\underline{G}}$-subring B of A and finitely generated B-submodule Y of M, B n-paralyzes (n-stabilizes) Y.

Then if $L \in L\underline{\underline{Q}}_n$ it follows that L locally n-paralyzes itself (under the adjoint action). Also if $A = L/L^2$ and M is a factor of the lower central series of L^2 then A locally n-paralyzes M and both A and M are abelian.

We shall show that if $L \in L\underline{\underline{Q}}_n$ then $L \in \underline{\underline{Q}}_{2n}$ in 2 stages:

(1) We show that the torsion ideal T of L is n-paralyzed by L.

(2) We assume that L is torsion-free and show that we may equivalently consider L to be a Lie algebra over a field. We let $A = L/L^2$ and look at the action of A on each factor M of the lower central series of L^2 and show

that A n^2-paralyzes M. Then since L^2 stabilizes its own lower central series, L n^3-paralyzes L^2 and since $L/L^2 \in \underline{\underline{A}}$ we deduce that $L \in \underline{\underline{Q}}_{n^3+1}$. Since also $L \in L\underline{\underline{Q}}_n$ we then show that $L \in \underline{\underline{Q}}_n$.

Returning to the general case when $T \neq 0$ it follows that $L \in \underline{\underline{Q}}_{2n}$.

Theorem 7.1

If T is a periodic L-module and L locally n-paralyzes T then L n-paralyzes T.

Proof

Let B be a $\underline{\underline{G}}$-subring of L and X a finitely generated B-submodule of T. Suppose $B = < b_i : 1 \leqslant i \leqslant s >$ and $X \doteq << y_1, \ldots, y_r >^B>$ then $X = \sum_{j=1}^{r} < y_j^B >$, then by proposition 4,1 $X \in \underline{\underline{T}}$. Since also X is periodic, $X \in \underline{\underline{F}}$. Hence there exist only finitely many series of X paralyzed by B. We denote the set of such series of length n by S(X,B). If B' is a $\underline{\underline{G}}$-subring of L and X' a finitely generated B'-submodule of T such that $B \leqslant B'$ and $X \leqslant X'$ we can define $\theta : S(X',B') \to S(X,B)$ by $\theta(Y_0, \ldots, Y_n) = (Y_0 \cap X, \ldots, Y_n \cap X)$ since this series is paralyzed by X. Thus the set of all S(X,B) form a projection set where $S(X,B) \leqslant S(X',B')$ if $B \leqslant B'$ and $X \leqslant X'$ and so there exists an element (K_{XB})

of the inverse limit (see [10]). Suppose $K_{XB} = (X_{o,B},\dots,X_{n,B})$ and let $T_i = \bigcup_{X,B} X_{i,B}$ for $0 \leqslant i \leqslant n$, where the union is taken over all $\underline{\underline{G}}$-subrings B of L and finitely generated B-submodules X of T. Now consider the series $(T_i : 0 \leqslant i \leqslant n)$ of T; if $t \in T_i$ for some $1 \leqslant i \leqslant n$ and $x \in L$ then $t \in X_{i,B}$ for some X and B and we may assume that $x \in B$. Therefore $tx = mt \bmod X_{i-1,B}$ for some $m \in Z$ and so $tx = mt \bmod T_{i-1}$. Hence the series $(T_i : 0 \leqslant i \leqslant n)$ is paralyzed by L, and so L n-paralyzes T.

<u>Corollary 7.1.1</u>

If L is periodic and $L \in L\underline{\underline{Q}}_n$ then $L \in \underline{\underline{Q}}_n$.

<u>Proof</u>

Put T = L in theorem 7.1.

We now consider the case when $L \in L\underline{\underline{Q}}_n$ and L is torsion-free. The next result shows why we may equivalently consider L to be a Lie algebra over a field.

<u>Proposition 7.2</u>

If A and M are torsion-free then A n-paralyzes M if and only if $A \otimes_Z Q$ n-paralyzes $M \otimes_Z Q$ as a Lie algebra over Q.

Proof

Let $(M_i : 0 \leqslant i \leqslant n)$ be a series of M paralyzed by A (we may assume M_i/M_{i-1} is torsion-free by lemma 2.5) and consider the series $(M_i \otimes_Z Q : 0 \leqslant i \leqslant n)$ ... (1) Let $x \otimes r \in M_i \otimes_Z Q$ for some $1 \leqslant i \leqslant n$ and $a \otimes q \in A \otimes_Z Q$, then $(x \otimes r)(a \otimes q) = xa \otimes rq$. Now $kx \in M_i$ for some $k \in Z^*$ so $kxa = mx \bmod M_{i-1}$ for some $m \in Z$. Therefore we have $xa \otimes rq = kxa \otimes rq = kxa \otimes k^{-1}rq = mx \otimes k^{-1}rq \bmod M_{i-1} \otimes_Z Q$ so $xa \otimes rq = mk^{-1}q(x \otimes r) \bmod M_{i-1} \otimes_Z Q$. Since $mk^{-1}q \in Q$ it follows that the series (1) is paralyzed by $A \otimes_Z Q$ as a Lie algebra over Q.

Conversely suppose $(V_i \otimes_Z Q : 0 \leqslant i \leqslant n)$ is a series of $M \otimes_Z Q$ paralyzed by $A \otimes_Z Q$ as a Lie algebra over Q. Consider the series $(W_i : 0 \leqslant i \leqslant n)$ where we define
$W_i = \{x \in M : \text{there exists } k \in Z^* \text{ such that } kx \in V_i\}$...(2)
Let $a \in A$, $y \in W_i$ for some $1 \leqslant i \leqslant n$ then $ky = x \in V_i$ for some $k \in Z^*$. Now $x \otimes 1 \in V_i \otimes_Z Q$ so there exist $t, r \in Q$ and $z \in V_{i-1}$ such that $(y \otimes 1)(a \otimes 1) = t(x \otimes 1) + z \otimes r$. Now there exists $q \in Z$ such that $qr = m \in Z$, then $(qx \otimes 1)(a \otimes 1) = t(qx \otimes 1) + mz \otimes 1$. Therefore
$tqx \otimes 1 = qxa \otimes 1 - mz \otimes 1 = (qxa - mz) \otimes 1 \in M \otimes_Z 1$ ...(3)

Now $t \in Q$ so $tqx \circ 1$ may be written as $uqx \circ s$ where $u, s \in Z$. By (3) there exists u such that $s = 1$ i.e. $t \in Z$. Also $q(tx - xa) \in V_{i-1}$ i.e. $qk(ty - ya) \in V_{i-1}$ and therefore by (2) $ty - ya \in W_{i-1}$ or $ya = ty \bmod W_{i-1}$. Since $t \in Z$, the series $(W_i : 0 \leq i \leq n)$ is paralyzed by A.

<u>The Scalar Function Set of a Series</u>

Let A and M be Lie algebras over a field F and suppose that A n-paralyzes M i.e. there exists a series of M $(M_i : 0 \leq i \leq n) \ldots (*)$ paralyzed by A. By lemma 1.6 A acts universally on each factor of the series, so if $1 \leq i \leq n$ and $M_{i-1} < M_i$ we define $g_i : A \to F$ such that if $x \in M_i$ then $[x, a] = g_i(a)x \bmod M_{i-1}$ for all $a \in A$. We denote the set of all g_i by $E_A(*)$. Thus the cardinality of this set may be less than n since g_i may be equal to g_j for some $j \neq i$ or g_i may not be defined for some i. In particular if $M = 0$ then $E_A(*) = \emptyset$.

<u>Lemma 7.3</u>

If $(V_i : 0 \leq i \leq n) \ldots (*)$ and $(W_j ; 0 \leq j \leq m) \ldots (**)$ are two series of M paralyzed by A where Mand A are Lie algebras over a field, then $E_A(*) = E_A(**)$.

Proof

W.l.o.g. we assume (*) and (**) are proper series (i.e. all terms are distinct). By proposition 2.1 there exist A-invariant isomorphic refinements $(X_i : 0 \leqslant i \leqslant t)$ and $(Y_j : 0 \leqslant j \leqslant t)$ of (*) and (**). Therefore there exists $\sigma \in S_t$ such that $X_i/X_{i-1} \cong Y_{\sigma(i)}/Y_{\sigma(i)-1}$ under the isomorphism θ_i for all $1 \leqslant i \leqslant t$. Again w.l.o.g we assume $X_{i-1} < X_i$ for all $1 \leqslant i \leqslant t$, then for $1 \leqslant k \leqslant n$ we define $f(k)$ to be the least integer for which $V_{k-1} < X_{f(k)}$. Let $\sigma(f(k)) = g(k)$ then we define $h(k)$ to be the least integer for which $Y_{g(k)} \leqslant W_{h(k)}$. Let $x \in X_{f(k)}$ and let $y = \theta_{f(k)}(x)$, then $y \in Y_{g(k)}$. Since A paralyzes the series (*) and (**), if $a \in A$, $\theta_{f(k)}(xa) = \theta_{f(k)}(x)a = ya = \phi(a)y \bmod Y_{g(k)-1}$ for some $\phi \subset E_A(**)$. But $xa = \psi(a)x \bmod X_{k-1}$ for some $\psi \in E_A(*)$, and therefore $\phi = \psi$. Hence $E_A(*) \leqslant E_A(**)$ and by symmetry $E_A(**) \leqslant E_A(*)$. Therefore we have equality.

The Scalar Function Set of a Module

If M and A are Lie algebras over a field and M is n-paralyzed by A we can define the scalar function set of M, $E_A(M)$ such that $E_A(*) = E_A(M)$ for any series (*) of M paralyzed by A, by lemma 7.3

Proposition 7.4

If A is an abelian Lie algebra over a field F which locally n-paralyzes the module M then A n^2-paralyzes M.

Proof

Let B be a $\underline{\underline{G}}$-subalgebra of A and V, W finitely generated B-submodules of M. If $(W_i : 0 \leqslant i \leqslant n)$ is a series of W paralyzed by B and $V \leqslant W$ then $(W_i \cap V : 0 \leqslant i \leqslant n)$ is a series of V paralyzed by B. Hence $E_B(V) \leqslant E_B(W)$ by lemma 7.3.
We therefore deduce that there exists a set E of functions from B to F such that $E_B(V) \leqslant E$ for all finitely generated B-submodules V of M, and $|E| = m \leqslant n$. So if $E = \{g_1, \ldots, g_m\}$ there exist sets $T_V = \{\sigma_1, \ldots, \sigma_k\}$ where $\sigma_i : \{1, \ldots n\} \rightarrow \{1, \ldots, m\}$ such that if $\sigma \in T_V$ there exists a series $(V_i : 0 \leqslant i \leqslant n)$ of V with the property that if $x \in V_i$ for some $1 \leqslant i \leqslant n$ then $xb = g_{\sigma(i)}(b)x \bmod V_{i-1}$ for all $b \in B$.

Clearly $T_{V+W} \leqslant T_V \cap T_W$ and $1 \leqslant |T_V| \leqslant n^m$ if $V \neq 0$. We now show there exists $\sigma \in \bigcap_{V \neq 0} T_V$, for suppose not then for each $\sigma : \{1, \ldots, n\} \rightarrow \{1, \ldots, m\}$ there exists $V_\sigma \neq 0$ with $\sigma \notin T_{V_\sigma}$. Then if $W = \sum_\sigma V_\sigma$, $T_W \leqslant \bigcap_\sigma T_{V_\sigma} = \emptyset$ – a contradiction.
We now choose any $\sigma \in \bigcap_{V \neq 0} T_V$ and define $h_i = g_{\sigma(i)}$ for $1 \leqslant i \leqslant n$. Then if V is any finitely generated B-submodule of M there exists a series $(V_i : 0 \leqslant i \leqslant n)$ of V such that if $x \in V_i$

where $1 \leqslant i \leqslant n$ and $b \in B$ then $xb = h_i(b) \bmod V_{i-1}$ ...(1)
Let $M_o = 0$ and inductively define for $1 \leqslant i \leqslant n$:
$M_i = \{x \in M : xb = h_i(b)x \bmod M_{i-1}$ for all $b \in B\}$. Clearly the series $\{M_i : 0 \leqslant i \leqslant n\}$ of M_n is (universally) paralyzed by B. Suppose there exists $x \in M \ M_n$ and $V = \langle x^B \rangle$, then from (1) it follows that $V_1 \leqslant M_1$ and if $V_i \leqslant M_i$ and $y \in V_{i+1}$ for some $1 \leqslant i \leqslant n-1$ then $yb = h_{i+1}(b)y \bmod V_i$, and so $yb = h_{i+1}(b)y \bmod M_i$. Therefore $V_n \leqslant M_n$ and $x \in M_n$, a contradiction. We deduce that $M = M_n$ and B n-paralyzes M.

Now h_i is a Lie homomorphism from B to F with kernal $C_i = C_B(M_i/M_{i-1})$. Hence $B/C_i \cong 0$ or F, so if we define $C_B = \bigcap_{i=1}^{n} C_i$ then B/C_B has dimension $\leqslant n$, i.e. lies in the (Lie algebra) class $\underline{\underline{T}}_n$. Furthermore C_B n-stabilizes M, so in the bracket notation we may write $[M,_n C_B] = 0$ ...(2)

Let $C = \langle C_B : B \leqslant A$ and $B \in \underline{\underline{G}} \rangle$ and suppose D is a $\underline{\underline{G}}$-subring of C. We may write $D = \langle c_1, \ldots, c_s \rangle$ where w.l.o.g. we assume $c_i \in C_{B_i}$ for $1 \leqslant i \leqslant s$. If $r \in Z^+$, $\langle [M,_r D] \rangle$ is generated by elements of the form $[v, y_1, \ldots, y_r]$ ...(3) where $y_i \in \{c_1, \ldots, c_s\}$ if $1 \leqslant i \leqslant s$ and $v \in M$. Now since $C \in \underline{\underline{A}}$ $[v, y_1, \ldots y_r] = [v, y_{\sigma(1)}, \ldots, y_{\sigma(r)}]$ for any $\sigma \in S_r$ by lemma 5.3. Hence if $r = (n-1)s+1$ then $[M,_r D] = 0$ since at least one c_i must occur n times in the expression (3).

Also, since $D \leqslant A$, D n-paralyzes M by the first part of the proof, so if $(V_i : 0 \leqslant i \leqslant n)$ is a series of M paralyzed by D and $x \in V_i$ for some $1 \leqslant i \leqslant n$ and $d \in D$ then there exists $t \in Z$ such that $[x,d] = tx \bmod V_{i-1}$. Therefore $0 = [x,_r d] = t^r x \bmod V_{i-1}$ so $t^r = t = 0$ and D stabilizes the series $(V_i ; 0 \leqslant i \leqslant n)$. Therefore $[M,_n D] = 0$ for all $\underline{\underline{G}}$-subrings D of C and so $[M,_n C] = 0$ i.e. C n-stabilizes M.

We now show that $A/C \in \underline{\underline{T}}_n$. Suppose not then there exists a subspace Y of A/C with dimension $n+1$. Let $\{b_i + C \; ; \; 1 \leqslant i \leqslant n+1\}$ be a basis of Y and set $B = \langle b_1, .., b_{n+1} \rangle$ We know from (2) that $B/C_B \in \underline{\underline{T}}_n$ and since $C_B \leqslant C$ this implies that $Y \in \underline{\underline{T}}_n$ - a contradiction. Therefore $A/C \in \underline{\underline{T}}_n$.

We may now suppose that $A = \langle B,C \rangle$ where $B \in \underline{\underline{T}}_n$ $B = \langle b_1, \dots, b_n \rangle$. Therefore there exists a series $(M_i : 0 \leqslant i \leqslant n)$ of B-submodules of M paralyzed by B, again by the first part of the proof. Also C stabilizes the series $(V_i : 0 \leqslant i \leqslant n)$ where $V_i = [M,_{n-i} C]$ for $0 \leqslant i \leqslant n$, and since $C \lhd A$, this series is A-invariant.

Now if $X = V_j$ and $Y = V_{j+1}$ for some $0 \leqslant j \leqslant n-1$ then $(((M_i \cap Y) + X)/X \; : 0 \leqslant i \leqslant n)$ is a series of Y/X paralyzed by B and stabilized by C, and so paralyzed by $A = B + C$ Thus A n^2-paralyzes M.

Thus if L is a Lie algebra over a field and $L \in L\underline{\underline{Q}}_n$ and $A = L/L^2$ then A n^2-paralyzes each factor of the lower central series of L^2. Since L^2 stabilizes this series, L n^3-paralyzes L^2, so $L \in \underline{\underline{Q}}_{n^3+1}$ since $L/L^2 \in \underline{\underline{A}}$. The next shows that since $L \in L\underline{\underline{Q}}_n \cap \underline{\underline{Q}}_{n^3+1}$, $L \in \underline{\underline{Q}}_n$.

<u>Proposition 7.5</u>

If A and M are Lie algebras over a field F and A s-paralyzes and locally n-paralyzes M then A n-paralyzes M.

<u>Proof</u>

W.l.o.g. we assume $n \leqslant s$. Suppose the series of M $(M_i : 0 \leqslant i \leqslant s)$...(1) is paralyzed by A and let $E_A(M) = \{f_1, \ldots, f_t\}$.

Let B be a $\underline{\underline{G}}$-subalgebra of A and V a finitely generated B-submodule of M. The series $(V \cap M_i : 0 \leqslant i \leqslant s)$ is paralyzed by B so by lemma 7.3 if $g \in E_B(V)$ then $g = f_i|_B$ for some $1 \leqslant i \leqslant t$. We choose B and V such that $E_B(V)$ has maximal cardinality m, and define an injective map $\theta : \{1, \ldots, m\} \to \{1, \ldots, t\}$ such that $g_i = f_{\theta(i)}|_B$. If C is a $\underline{\underline{G}}$-subalgebra of A and W a finitely generated C-submodule of M then $D = \langle B, C \rangle$ is a $\underline{\underline{G}}$-subalgebra of A and $U = \langle (W,V)^D \rangle$ is a finitely generated D-submodule of M. Then $E_D(U)$ also has cardinality m, and again we define $\phi : \{1,\ldots,m\} \to \{1,\ldots,t\}$ such that if $h \in E_D(U)$ then $h = f_{\phi(i)}|_D$. Now if $(U_i : 0 \leq i \leq n)$ is a series of U paralyzed by D then $(U_i \cap V : 0 \leq i \leq n)$ is a series of V paralyzed by B. Therefore if $g \in E_B(V)$ then $g = h|_B$ for some $h \in E_D(U)$ i.e. $g = f_{\phi(i)}|_B$ for some $1 \leq i \leq m$. It therefore follows that $\theta = \phi$.

Since also $(U_i \cap W : 0 \leq i \leq n)$ is a series of W paralyzed by C, it follows that $E_C(W) \leq \{f_{\theta(i)}|_C : 1 \leq i \leq m\}$. As in the proof of proposition 7.4, there exists $\sigma : \{1,\ldots,n\} \to \{1,\ldots,m\}$ such that if C is any $\underline{\underline{G}}$-subalgebra of A and W any finitely generated C-submodule of M then there exists a series $(W_i : 0 \leq i \leq n)$ of W such that if $x \in W_i$ for some $1 \leq i \leq n$ then $xa = [x,a] = k_i(a)x \bmod W_{i-1}$ for all $a \in C$ and where $k_i = f_{\sigma(\theta(i))}$.

We now set $M_o = 0$ and inductively define for $1 \leq i \leq n$ $M_i = \{x \in M : [x,a] = k_i(a)x \bmod M_{i-1}$ for all $a \in A\}$. Then clearly $(M_i : 0 \leq i \leq n)$ is a series of M_n paralyzed by A. Suppose there exists $x_1 \in M \setminus M_n$, then there exists $a_1 \in A$ such that $[x_1,a_1] \neq k_n(a_1)x_1 \bmod M_{n-1}$; and therefore $x_2 = [x_1,a_1] - k_n(a_1)x_1 \notin M_{n-1}$. Inductively we define $a_2, x_3, a_3, \ldots, a_n x_{n+1}$ such that: for $1 \leq i \leq n$ $x_{i+1} = [x_i,a_i] - k_{n-i+1}(a_i) \notin M_{n-i}$.

Let $B = \langle a_1,\ldots,a_n \rangle$ and V the B-submodule of M generated by $x_1,\ldots,x_n$ i.e. $V = \langle\langle x_1,\ldots,x_n \rangle^B\rangle$. therefore there exists a series $(V_i : 0 \leq i \leq n)$ of V paralyzed by B and such that if $x \in V_i$ for some $1 \leq i \leq n$ then $[x,a] = k_i(a)x \bmod V_{i-1}$ for all $a \in B$. Now $x_1 \in V_n$ so $[x_1,a_1] = k_n(a_1)x_1 \bmod V_{n-1}$ and so $x_2 \in V_{n-1}$. Similarly $x_{i+1} \in V_{n-i}$ for $1 \leq i \leq n-1$ and in particular $[x_n,a_n] = k_1(a_n)x_n$ i.e. $x_{n+1} = 0$ - a contradiction. Therefore $M = M_n$ and A n-paralyzes M.

Corollary 7.5.1

If $\underline{\underline{X}}$ is the class of torsion-free Lie rings then $L\underline{\underline{Q}}_n \cap \underline{\underline{X}} \leq \underline{\underline{Q}}_n$.

Proof

Let $L \in L\underline{\underline{Q}}_n \cap \underline{\underline{X}}$ and form $\bar{L} = L \otimes_Z Q$. By propositions 7.2 and 7.4 $\bar{L}$ lies in the class of Lie algebras $\underline{\underline{Q}}_{n^3+1}$. Now by proposition 7.5 with $s = n^3+1$ and $A = M = \bar{L}$ it follows that $\bar{L} \in \underline{\underline{Q}}_n$. Finally by proposition 7.2, $L \in \underline{\underline{Q}}_n$.

Note also that for Lie algebras over any field, $L \in L\underline{\underline{Q}}_n$ implies that $L \in \underline{\underline{Q}}_n$.

Theorem 7.6

$$L\underline{\underline{Q}}_n \leqslant \underline{\underline{Q}}_{2n}$$

Proof

Let $L \in L\underline{\underline{Q}}_n$ and let T be the torsion ideal of L. By theorem 7.1, T is n-paralyzed by L. L/T is torsion-free and since $QL\underline{\underline{Q}}_n = L\underline{\underline{Q}}_n$, $L/T \in L\underline{\underline{Q}}_n$ so by corollary 7.5.1, $L/T \in \underline{\underline{Q}}_n$. Therefore L has a quasicentral series of length 2n, i.e. $L \in \underline{\underline{Q}}_{2n}$.

CHAPTER 8

In this chapter we apply some of the techniques used in the preceeding chapters to obtain corresponding group theory results. We shall use the standard notation for derived groups, centralizers, normalizers, and automorphisms etc. and now $\underline{\underline{A}}, \underline{\underline{G}}, \underline{\underline{N}}, \underline{\underline{S}}$ denote the classes of abelian, finitely generated, nilpotent and supersoluble groups respectively. The closure operations L,P,Q,R,S are defined on classes of groups in the usual way (see [5]).

Paralyzers and Stabilizers

Let G be a group and $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ ...(*) a series of G. We define the stabilizer of (*) (the stability group of [6]) to be $\bigcap_{\sigma\in\Sigma} C_A(\Lambda_\sigma/V_\sigma)$ where $A = \mathrm{Aut}(G)$. Therefore if $x \in \Lambda_\sigma$ and $a \in A$ then

$$x^a = x \bmod V_\sigma \qquad \ldots(1)$$

equivalently $\quad [x,a] \in V_\sigma \qquad \ldots(2)$

where $[x,a]$ denotes $x^{-1}x^a$.

If $X,Y \leqslant G$ we define the quasicentralizer of Y in X, $K_X(Y) = \bigcap_{y\in Y} N_X(\langle y\rangle)$. We do not have as strong a result as lemma 1.5 for groups:

Lemma 8.1

If $Y \leqslant N_G(X)$ then $K_Y(X) \trianglelefteq Y$.

Proof

Let $K = K_Y(X)$ and $k \in K$, $y \in Y$ and $x \in X$. Then $x^{k^y} = y^{-1}k^{-1}yxy^{-1}ky$; but $yxy^{-1} = z \in X$ since $y \in N_G(X)$ so $k^{-1}zk = z^n$ for some $n \in Z$ and $x^{k^y} = (z^n)^y = (z^y)^n = x^n$. Hence $k^y \in K$ and therefore $K \lhd Y$.

Automorphisms which fix every subgroup of a group G are called power automorphisms ([6] p209). A power automorphism θ is universal on G if there exists $n \in Z$ such that $g^\theta = g^n$ for all $g \in G$ (see [2]). Power automorphisms are universal on groups in $\underline{\underline{G}} \cap \underline{\underline{A}}$ ([6] p 210). We have the following special case of lemma 8.1:

Lemma 8.1.1

If $X \leqslant G$ and $X \in \underline{\underline{A}}$, then $[N_G(X), K_G(X)] \leqslant C_G(X)$.

Proof

Let $x \in X$, $g \in N_G(X)$ and $k \in K_G(X)$, then if $x^k = x^n$, $x^{kg} = (x^n)^g = (x^g)^n = x^{gk}$ since conjugation by k acts as a power automorphism of $< x, x^g > \in \underline{\underline{G}} \cap \underline{\underline{A}}$. Thus $[g,k] \in C_G(X)$, and $[N_G(X), K_G(X)] \leqslant C_G(X)$.

That $X \in \underline{\underline{A}}$ is necessary in lemma 8.1.1 is shewn by the following example:

Example

Let $X = Q_8$ the quaternion group; we write $X = \{ \pm 1, \pm i, \pm j, \pm k : i^2 = j^2 = k^2 = -1, ij = k \}$. Let $A = \mathrm{Aut}(X)$ and $G = XA$ the semidirect product, then $X \lhd G$. Let $a \in A$ be defined by $i^a = -i$, $j^a = j$, and $k^a = -k$, then since $-i = i^3$, so $a \in K_G(X)$. If $b \in A$ is defined by $i^b = j$, $j^b = k$, $k^b = i$, then $i^{ba} = j$ but $i^{ab} = -j$. Thus $ba \notin C_G(X)$ and $[N_G(X), K_G(X)] \nleq C_G(X)$.

We define the paralyzer of (*) to be $\bigcap_{\sigma \in \Sigma} K_A(\Lambda_\sigma / V_\sigma)$ then if $|\Sigma| < \infty$ this is the 'paralyzing group of automorphisms' of [8]. Now suppose H is a set of operators on G, then we shall say that H paralyzes (stabilizes) (*) if $H/C_H(G)$ embeds in the paralyzer (stabilizer) of (*), and does so faithfully if $C_H(G) = 1$. We shall also say that H n-paralyzes (n-stabilizes) G if there exists a series (*) of G with $|\Sigma| = n$, paralyzed (stabilized) by H.

If $G/Z(G)$ embeds in the paralyzer of (*) we shall call (*) a quasicentral series. If we now define $\underline{\underline{Q}}_n$ to be the class of groups with a quasicentral series of length $\leqslant n$, then $\underline{\underline{Q}}_n$ is not the class $\underline{\underline{P}}_n$ of parasoluble groups of paraheight $\leqslant n$ of Hill [8], since Hill's definition requires an abelian quasicentral series of length $\leqslant n$. However, since each factor of a quasicentral series is Dedekind and therefore a member of $\underline{\underline{N}}_2$ (see e.g. [12]) by insertion of the derived groups of the members of a quasicentral series of length $\leqslant n$ we achieve a 'parasoluble' series of length $\leqslant 2n$. Thus $\underline{\underline{Q}}_n = \underline{\underline{P}}_{2n}$.

Proposition 8.2

If $(\Lambda_\sigma, V_\sigma : \sigma \in \Sigma)$ is an invariant series of G with paralyzer A then $[G,A] \in \underline{\underline{Q}}_\Sigma$.

Proof

Let $H = [G,A]$ and P the semidirect product GA. Let $K = K_P(\Lambda_\sigma/V_\sigma)$ then $K_\sigma \lhd P$ for all $\sigma \in \Sigma$ by lemma 8.1, and $A \leqslant \bigcap_{\sigma\in\Sigma} K_\sigma$. Therefore $H \leqslant \bigcap_{\sigma\in\Sigma} K_\sigma$ and it follows that $(H \cap \Lambda_\sigma, H \cap V_\sigma : \sigma \in \Sigma)$ is a quasicentral series of H. Note also that if $|\Sigma| = n$ then $[G,A] \in \underline{\underline{P}}_{2n}$ as shewn in [8].

The methods of chapter 3 cannot be extended to group theory, since they make use of the linear structure of Lie rings. However, in chapter 4, with more restrictive conditions and slight modifications to the proofs, we can obtain some corresponding results. Recall from chapter 1 and chapter 2 that if L is a Lie ring, we can construct an abelian Lie ring L^* group-isomorphic to L, and that the structure of the paralyzer of a series of L did not depend on the invariance of the series, since it also paralyzed the corresponding series of L^*. Since we do not have this property in group theory, the following result is a weak analogue of theorem 4.4.

Proposition 8.3

If A faithfully paralyzes the ascending cyclic invariant series $(V_\sigma : \sigma \leqslant \rho)$ of G then $A \in L R \underline{\underline{S}}$.

Proof

We begin as in theorem 4.4; let B be a $\underline{\underline{G}}$-subgroup of A and $b \in B$. Then there exists $x \in G$ such that $x^b \neq x$ so let $X = \langle x^B \rangle$ and $C = C_B(X)$ then $C \lhd B$, $b \notin C$ and B/C faithfully paralyzes the ascending cyclic invariant series $(V_\sigma \cap X : \sigma \leqslant \rho)$ of X. By a similar method to that of proposition 4.1 we can shew that $X \in \underline{\underline{G}}$. Now since G is hypercyclic and therefore locally supersoluble, $X \in \underline{\underline{S}}$ and satisfies the maximal condition on subgroups. Hence the series $(V_\sigma \cap X : \sigma \leqslant \rho)$ has only finitely many distinct terms. By theorem 1 of [3.], $B/C \in \underline{\underline{S}}$, and so $A \in L R \underline{\underline{S}}$.

We can also shew that A has a descending series with A-hypercyclic factors, by a similar method to that of theorem 4.6.

Local Parasolubility

Let $G \in L\underline{\underline{P}}_n$; Hill has shewn in [8] that $G \in \underline{\underline{P}}_{f(n)}$ where $f(n) = (3.2^{n-1} - 1 - n)n + 1$. By similar methods to those used in chapter 7 we can improve this result to shew that $G \in \underline{\underline{P}}_{2n+1}$. We need the following definition:

If G is any group and A a set of operators on G then A locally n-paralyzes G if given X and B, finitely generated subgroups of G and A respectively, then B n-paralyzes X^B.

Thus if $G \in L\underline{\underline{P}}_n$, G locally n-paralyzes itself (under the action $x^g = g^{-1}xg$). Also $G \in L(\underline{\underline{N}}_n\underline{\underline{A}}) = \underline{\underline{N}}_n\underline{\underline{A}}$ so if $A = G/G'$ then A locally n-paralyzes each factor M of the lower central series of G', and $A, M \in \underline{\underline{A}}$. Recall from chapter 7 that given a locally parasoluble Lie ring, we first factored by the torsion ideal which left a torsion-free Lie ring which we shewed could equivalently be regarded as a Lie algebra over Q. However the periodic elements of G may not generate a periodic normal subgroup (unless G has no elements of even order (see [19]) - a special case to which we shall return later). We do know that the torsion elements of G' form a subgroup, since $G' \in \underline{\underline{N}}$ (see [14] p181). If we denote this subgroup by T, a slight adaptation of theorem 7.1 shews that T has an abelian series of length n paralyzed by G. We shew first that if X is a $\underline{\underline{G}}$-subgroup of T and B a $\underline{\underline{G}}$-subgroup of G then $X^B \in \underline{\underline{G}}$; this follows from a similar argument to that given in proposition 4.1, then since $G \in L\underline{\underline{S}}$, $|X^B| < \infty$ and we can continue as in theorem 7.1.

We next therefore consider the case when G' is torsion-free. Then each factor M of the lower central series is torsion-free. We write M additively and A multiplicatively and form the group ring ZA, which is a commutative ring. Then if $a \in A$, ma is the map such that $x^{ma} = mx^a$. Since M is abelian, ZA is a ring of operators of M and clearly ZA also locally n-paralyzes M. If $b \in ZA$ and $kb = 0$ for some $k \in Z^+$ then $kM^b = 0$ so $b = 0$. Hence ZA is torsion-free (additively) and we can embed ZA in $ZA \otimes_Z Q$ and M in $M \otimes_Z Q$ and proceed as in propopopostion 7.4 to deduce that ZA and hence A n^2-paralyzes M. Hence there exists a series $(V_i : 0 \leqslant i \leqslant n^3)$ of G' with abelian factors, paralyzed by A and stabilized by G' and so paralyzed by G. By an extension of lemma 2.4 we can assume each factor of this series is torsion-free. We also know that if V and B are $\underline{\underline{G}}$-subgroups of G' and G respectively then there exists a series $(W_i : 0 \leqslant i \leqslant n)$ of V^B with (torsion-free) abelian factors, paralyzed by B. We can therefore define scalar function sets for V and B and continue in a method similar to that of proposition 7.5 to shew that G paralyzes an abelian series of length n of G'. Since also $G/G' \in \underline{\underline{A}}$, $G \in \underline{\underline{P}}_{n+1}$.

In the general case therefore, this outlines the proof of :

<u>Theorem 8.4</u>

$$L\underline{\underline{P}}_n \leqslant \underline{\underline{P}}_{2n+1}$$

We state the following special cases in:

<u>Theorem 8.5</u>

Let $G \in L\underline{\underline{P}}_n$,

(1) if G is periodic then $G \in \underline{\underline{P}}_n$

(2) if G' is torsion-free then $G \in \underline{\underline{P}}_{n+1}$

(3) if G is torsion-free then $G \in \underline{\underline{P}}_n$

(4) if G has no elements of even order then $G \in \underline{\underline{P}}_{2n}$

<u>Proof</u>

(1) As for theorem 8.4 with T = G.

(2) As for theorem 8.4 with T = 1.

(3) We know in this case, from the proof of theorem 8.4 that G paralyzes an abelian series of length n^3 of G', and therefore a torsion-free abelian series of length $n^3 + 1$ of itself. Using again scalar function sets and this time the fact that G locally n-paralyzes itself, we deduce that G n-paralyzes itself, i.e. $G \in \underline{\underline{P}}_n$.

(4) By Zappa [19] the elements of odd order of a supersoluble group form a subgroup. Since $G \in L\underline{\underline{S}}$, the torsion elements of G form a subgroup T. As in the proof of theorem 8.4 this is n-paralyzed by G; also G/T is torsion-free so by (3) $G/T \in \underline{\underline{P}}_n$. Hence $G \in \underline{\underline{P}}_{2n}$.

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